<u>Sequences and Series</u>

2.1 <u>Sequences</u>

2.1.1 <u>Definition of a sequence</u>

<u>Examples</u>:

(1) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots, \frac{1}{n}, \ldots\ldots$

(2) 1, -1, 1, -1, ……………

(3) 3, 5, 7, 9, 11, …………

(4) $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\ldots, \frac{1}{2^{n-1}}, \ldots$

(5) 0, 1, 0, 2, 0, 3, ……….

In each of the above we have an example of a sequence. In each of them we have an <u>endless list of numbers</u> and these numbers are <u>listed in order</u>.
Let us call these numbers the <u>terms</u> of the sequence.

<u>In example (1)</u>: The <u>first</u> term is 1, the <u>second</u> term is $\frac{1}{2}$, the <u>third</u> term is $\frac{1}{3}$ and so on.

For any given positive integer n, the <u>n^{th} term</u> is $\frac{1}{n}$. This is called the <u>general n^{th} term</u> and we <u>denote the sequence by</u> $\left\langle \frac{1}{n} \right\rangle$.

By just putting the values 1, 2, 3 etc for n in $\frac{1}{n}$, we get that the first term is 1, the second term is $\frac{1}{2}$, the third term is $\frac{1}{3}$ etc.

<u>In example (2)</u>: The first term is 1, the second term is -1, the third term is 1, the fourth term is -1 and so on. i.e., <u>we know for any positive integer n what the n^{th} term would be</u>. If n is odd it will be 1 and if n is even it will be -1.
We can express the n^{th} term as $(-1)^{n+1}$ since $(-1)^{n+1} = 1$ when n is odd and $(-1)^{n+1} = -1$ when n is even. So the sequence is $\langle (-1)^{n+1} \rangle$.

In example (3): The general n^{th} term is $2n + 1$. When $n = 1$, $2n + 1 = 3$ and when n increases by 1, $2n + 1$ increases by 2. So we get that, when $n = 2$, $2n + 1 = 5$, when $n = 3$, $2n + 1 = 7$ etc. So, the sequence is $\langle 2n+1 \rangle$.

In example (4): The sequence is $\left\langle \frac{1}{2^{n-1}} \right\rangle$.

In example (5): Although we have not given the general n^{th} term, we know that given any value for n, we can get the value of the n^{th} term. We proceed along 0, 1, 0, 2, 0, 3, and so on until we come to the n^{th} term.

Let us get the 16^{th} term in this way:
We have, 0, 1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, 0, 7, 0, 8.
The 16^{th} term is 8.

We see that when n is even, then n^{th} term is $\frac{n}{2}$ and when n is odd, the n^{th} term is 0.

When speaking about a sequence in general, we denote it by $\langle a_n \rangle$ (or $\langle x_n \rangle$, $\langle b_n \rangle$ etc.) and a_n denotes the general n^{th} term.

We can express the sequence in example 5 by $\langle a_n \rangle$ where $a_n = \frac{n}{2}$ when n is even and $a_n = 0$ when n is odd.

Example (6): For any positive integer n, $a_{n+1} = \frac{1}{1+a_n}$.

The above equation gives a sequence $\langle a_n \rangle$ once the value of a_1 is given.

(i) Let us take it that $a_1 = 0$. Then

$$a_2 = \frac{1}{1+a_1} = \frac{1}{1+0} = 1, \quad a_3 = \frac{1}{1+a_2} = \frac{1}{2}, \quad a_4 = \frac{1}{1+a_3} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3} \text{ etc.}$$

(ii) Let us take it that $a_1 = 1$. Then $a_2 = \frac{1}{1+a_1} = \frac{1}{2}$, $a_3 = \frac{1}{1+a_2} = \frac{2}{3}$ etc.

(iii) Let us take it that $a_1 = \alpha$ where $\alpha = \dfrac{\sqrt{5}-1}{2}$.

Now $1 + \alpha = 1 + \dfrac{\sqrt{5}-1}{2} = \dfrac{\sqrt{5}+1}{2}$ and

$$\frac{1}{1+\alpha} = \frac{2}{\sqrt{5}+1} = \frac{2(\sqrt{5}-1)}{5-1} = \frac{\sqrt{5}-1}{2} \text{ . i.e., } \frac{1}{1+\alpha} = \alpha .$$

So, $a_2 = \dfrac{1}{1+a_1} = \dfrac{1}{1+\alpha} = \alpha$, $\quad a_3 = \dfrac{1}{1+a_2} = \dfrac{1}{1+\alpha} = \alpha$ etc,

That is, for any positive integer n, $a_n = \alpha$.

We say here that $\langle a_n \rangle$ is a constant sequence since all the terms take the same value.

Example (7): $x_{n+1} = x_n + x_{n-1}$ for all $n \geq 2$.

The above equation gives a sequence $\langle x_n \rangle$ once the values of x_1 and x_2 are given.

2.1.2 Convergent and Divergent Sequences

Examples:

(1) Consider the sequence $\left\langle \dfrac{1}{n} \right\rangle$. i.e., $1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \ldots\ldots$. We see that as n grows indefinitely large, the value of $\dfrac{1}{n}$ approaches the value 0. We say that the sequence $\left\langle \dfrac{1}{n} \right\rangle$ converges to 0 as n tends to infinity and we write $\lim_{n\to\infty} \dfrac{1}{n} = 0$.

(2) Consider the sequence $\left\langle \dfrac{n}{n+1} \right\rangle$. i.e., $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \dfrac{5}{6}, \ldots\ldots$

We see that, $\lim_{n\to\infty} \dfrac{n}{n+1} = 1$.

In examples (1) and (2) we have what we call convergent sequences.

(3) Consider the sequence $\langle (-1)^n \rangle$. i.e., -1, 1, -1, 1, -1, 1, ….

There is no number l such that $(-1)^n$ approaches the value l as n becomes indefinitely large. For this reason we say that $\langle (-1)^n \rangle$ is a divergent sequence.

(4) Consider the sequence $\langle 3n-7 \rangle$. i.e., -4, -1, 2, 5, 8, 11, 14, …….
The is no number l such that $3n - 7$ approaches the value l as n becomes indefinitely large. So, $\langle 3n-7 \rangle$ is a divergent sequence.

2.1.3. Limits of a sequence

Definition 1: Suppose $\langle a_n \rangle$ is a sequences and l is a real number. We say that, $\langle a_n \rangle$ converges to l and write $\lim_{n\to\infty} a_n = l$ if given any real number ε such that $\varepsilon > 0$, there is a positive integer n_0 such that, whenever $n > n_0$, $|a_n - l| < \varepsilon$.

(Note: This is the technical way of saying that a_n approaches the value l as n becomes indefinitely large).

In this case we also say that the sequence $\langle a_n \rangle$ is convergent.

When for a sequence $\langle a_n \rangle$ there is no number l such that $\lim_{n\to\infty} a_n = l$, i.e., when $\langle a_n \rangle$ is not convergent, we say that it is divergent.

Let us apply the definition to the sequence $\left\langle \frac{n}{n+1} \right\rangle$.

Let $\varepsilon > 0$ and $n \in N$.

$$\left| \frac{n}{n+1} - 1 \right| = \left| -\frac{1}{n+1} \right| = \frac{1}{n+1}.$$

$$\frac{1}{n+1} < \varepsilon \quad \text{if} \quad n+1 > \frac{1}{\varepsilon}. \text{ i.e., if } n > \frac{1}{\varepsilon} - 1.$$

We can find $n_0 \in N$ such that $n_0 > \frac{1}{\varepsilon} - 1$.

So for all n, $\left| \frac{n}{n+1} - 1 \right| < \varepsilon$ when $n > n_0$. Therefore, $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Definition 2: Suppose $\langle x_n \rangle$ is a sequence.

(i) We say that $\langle x_n \rangle$ diverges to infinity and write $\lim_{n\to\infty} x_n = \infty$, if given any real number k, there is a positive integer n_0 such that when $n > n_0$, $x_n > k$.

(Note: This is a technical way of saying that x_n grows indefinitely large as n becomes indefinitely large.)

(ii) We say that $\langle x_n \rangle$ diverges to minus infinity and write $\lim_{n\to\infty} x_n = -\infty$, if given any real number k, there is a positive integer n_0 such that when $n > n_0$, $x_n < k$.

(Note: This is a technical way of saying that x_n grows indefinitely small as n becomes indefinitely large. (note: $-\frac{1}{2}$ is smaller than $-\frac{1}{3}$, -100 is smaller than -2, etc.))

Let us apply definition 2(i) to the sequence $\langle n^2 \rangle$.
Let $k \in R$ and $n \in N$.
$n^2 \geq n$. Therefore, $n^2 > k$ when $n > k$.
We can find n_0 such that $n_0 > k$.
So, for all n, $n^2 > k$ when $n > n_0$.
Therefore, $\lim_{n\to\infty} n^2 = \infty$.

2.1.4. Elementary Properties of Limits

Suppose $\langle a_n \rangle, \langle b_n \rangle$ are sequences and $c \in R$. Then, $\langle ca_n \rangle$ denotes the sequence whose n^{th} term is ca_n, $\langle a_n + b_n \rangle$ denotes the sequence whose n^{th} term is $a_n + b_n$ etc. Like this, given two sequences, we can form other sequences by subtracting, multiplying, dividing. In this spirit, $\langle |a_n| \rangle$ denotes the sequence whose n^{th} term is $|a_n|$.

Theorem 1 (Algebra of Limits):

Suppose $\langle x_n \rangle, \langle y_n \rangle$ are convergent sequences and $\lim_n x_n = l_1$ and $\lim_n y_n = l_2$. Suppose $k \in R$ and $\langle a_n \rangle$ is the constant sequence where for all $n \in N$, $a_n = k$. Suppose $c \in R$.
Then:

(i) $\lim_{n\to\infty} a_n = k$. (i.e., $\lim_{n\to\infty} k = k$)

(ii) $\lim_{n\to\infty}(cx_n) = cl_1$. (i.e., $\lim_{n\to\infty}(cx_n) = c\lim_{n\to\infty} x_n$)

(iii) $\lim_{n\to\infty}(x_n + y_n) = l_1 + l_2$. (i.e., $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$)

(iv) $\lim_{n\to\infty}(x_n - y_n) = l_1 - l_2$. (i.e., $\lim_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n$)

(v) $\lim_{n\to\infty}(x_n . y_n) = l_1 . l_2$. (i.e., $\lim_{n\to\infty}(x_n . y_n) = (\lim_{n\to\infty} x_n) \times (\lim_{n\to\infty} y_n)$)

(vi) $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{l_1}{l_2}$ (i.e., $\lim_{n\to\infty}(\frac{x_n}{y_n}) = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$), provided that $l_2 \neq 0$ and $y_n \neq 0$ for all $n \in N$.

We also have

(vii) $\lim_{n\to\infty}|x_n| = |l_1|$. (i.e., $\lim_{n\to\infty}|x_n| = |\lim_{n\to\infty} x_n|$)

Using definition 1, we can easily prove that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Now, by applying the theorem we get

$\lim_{n\to\infty}(\frac{1}{n^2}) = 0$, $\lim_{n\to\infty}(\frac{1}{n^3}) = 0$, $\lim_{n\to\infty}(\frac{1}{n^2} - \frac{1}{n^3}) = 0$, etc.

Consider for example, $\lim_{n\to\infty} a_n$ where $a_n = \frac{n^3 - 3n^2 + 1}{-4n^3 + 5}$.

Then, $a_n = \frac{1 - \frac{3}{n} + \frac{1}{n^3}}{-4 + \frac{5}{n^3}}$ and we see that by repeatedly applying the theorem we get, $\lim_{n\to\infty} a_n = \frac{1}{-4} = -\frac{1}{4}$.

The limits in the theorem are finite limits. Definition 1 is about finite limits. In definition 2 we have infinite limits. (i.e., $\lim_{n\to\infty} a_n = \infty$, $\lim_{n\to\infty} b_n = -\infty$)

What about the properties of infinite limits?

For instance, we have that if $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} y_n = \infty$, then $\lim_{n\to\infty}(x_n + y_n) = \infty$.

This can be coded as $\infty + \infty = \infty$.

We now give some properties of infinite limits in code.

Theorem 2:

(1) $\infty + \infty = \infty$
(2) $\infty \times \infty = \infty$
(3) $\infty \times (-\infty) = -\infty$
(4) $(-\infty) \times (-\infty) = \infty$
(5) $\frac{1}{\infty} = 0$ and $\frac{1}{-\infty} = 0$
(6) Suppose $l \in R$ and l is a constant. Then, $\frac{l}{\infty} = 0$ and $\infty + l = \infty$ and $-\infty + l = -\infty$.
(7) Suppose $l \in R$ and l is a constant. Then:
  (i) If $l > 0$, $l \times \infty = \infty$ and $l \times (-\infty) = -\infty$
  (ii) If $l < 0$, $l \times \infty = -\infty$ and $l \times (-\infty) = \infty$

Part of (6) (i.e., $\frac{l}{\infty} = 0$) decoded is: If $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} b_n = \infty$, then $\lim_{n\to\infty}(\frac{a_n}{b_n}) = 0$.

Example: Consider $\lim_{n\to\infty}(-n^5 + n^3)$.

$-n^5 + n^3 = n^3(-n^2 + 1)$ and $\lim_{n\to\infty} n^3 = \infty$ and $\lim_{n\to\infty}(-n^2 + 1) = -\infty$.

Therefore $\lim_{n\to\infty}(-n^5 + n^3) = -\infty$.

We can get this answer in the following way also: $-n^5 + n^3 = n^5(\frac{1}{n^2} - 1)$ and $\lim_{n\to\infty} n^5 = \infty$ and $\lim_{n\to\infty}(\frac{1}{n^2} - 1) = -1$. Therefore, $\lim_{n\to\infty}(-n^5 + n^3) = -\infty$.

Theorem 3 (Squeeze Rule):
Suppose $\langle a_n \rangle, \langle b_n \rangle, \langle c_n \rangle$ are sequences and $n_0 \in N$. Suppose for all n such that $n \geq n_0$, $a_n \leq c_n \leq b_n$. Now if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = l$ for some $l \in R$, then $\lim_{n\to\infty} c_n = l$.

Example: $0 \leq \sin\frac{1}{n} \leq \frac{1}{n}$ for all positive integers n (i.e., we can take the n_0 in the theorem as 1).

$\lim_{n\to\infty} 0 = \lim_{n\to\infty}\frac{1}{n} = 0$. Therefore, $\lim_{n\to\infty}(\sin\frac{1}{n}) = 0$.

Theorem 4:
Suppose $\langle a_n \rangle$ is a sequence and $n_0 \in N$ and $\lim_{n\to\infty} a_n = l$ for some $l \in R$. Suppose $k \in R$.
Then:
  (i) If $a_n \leq k$ for all n such that $n \geq n_0$, then, $l \leq k$.

(ii) If $a_n \geq k$ for all n such that $n \geq n_0$, then, $l \geq k$.

Example: Consider the sequence $\langle a_n \rangle$ where for all n, $a_n = 1 + (\frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!})$ (i.e.,

$a_1 = 1 + 1$, $a_2 = 1 + 1 + \frac{1}{2}$, $a_3 = 1 + 1 + \frac{1}{2} + \frac{1}{6}$ etc.)

It is known that $\langle a_n \rangle$ is convergent and $\lim_{n\to\infty} a_n = e$.

Now for all n such that $n \geq 2$, $a_n \geq 1 + 1 + \frac{1}{2} = 2.5$.

Therefore, $e \geq 2.5$

Exercise: Show that $e \geq 2.65$.

We give here some standard limits:

(1) Suppose r is a real number and r is a constant.

(i) If $|r| < 1$ (i.e., $-1 < r < 1$), then $\lim_{n\to\infty} r^n = 0$

(ii) If $r > 1$, then $\lim_{n\to\infty} r^n = \infty$

(2) Suppose a is a real number and $a > 0$ and a is a constant. Then $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$

(3) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

We end this section by giving a result that could be useful.

Result: Suppose $\langle a_n \rangle$ is a sequence. Then, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} |a_n| = 0$.

Example: Let $a_n = \frac{(-1)^n}{n}$ for all $n \in N$. Then, $|a_n| = \frac{1}{n}$ and hence $\lim_{n\to\infty} |a_n| = 0$.

Therefore, $\lim_{n\to\infty} a_n = 0$. i.e., $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

2.1.5. Monotonic Sequences

Consider the sequence $\langle a_n \rangle$ which is 1, 1, $\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}$, ...

We see that $a_1 \geq a_2 \geq a_3 \geq a_4 \geq$ ......

i.e., for all n, $a_n \geq a_{n+1}$. The terms of this sequence are non-increasing.

Now consider the sequence $\langle b_n \rangle$ which is 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ...

We see that $b_1 > b_2 > b_3 > \ldots\ldots$

i.e., for all n, $b_n > b_{n+1}$. The terms of this sequence are <u>decreasing</u>.

<u>Definition 1</u>: Suppose $\langle a_n \rangle$ is a sequence. <u>Then</u>:

(1) (a) If for all n, $a_n \geq a_{n+1}$,(i.e., $a_1 \geq a_2 \geq a_3 \geq a_4 \geq \ldots\ldots$) we say that $\langle a_n \rangle$ is monotonic decreasing (m.d).

(b) If for all n, $a_n > a_{n+1}$ (i.e., $a_1 > a_2 > a_3 > \ldots\ldots$), we say that $\langle a_n \rangle$ is <u>strictly monotonic decreasing.</u>

(2) (a) If for all n, $a_{n+1} \geq a_n$ (i.e., $a_1 \leq a_2 \leq a_3 \leq a_4 \leq \ldots\ldots$) we say that $\langle a_n \rangle$ is monotonic increasing (m.i).

(b) If for all n, $a_{n+1} > a_n$ (i.e., $a_1 < a_2 < a_3 < \ldots\ldots$), we say that $\langle a_n \rangle$ is <u>strictly monotonic increasing.</u>

<u>Note:</u> For real numbers x, y, $x \geq y$ means that $x > y$ or $x = y$. So, we see that if $\langle a_n \rangle$ is <u>strictly m.i</u> it is also <u>m.i</u> and if $\langle a_n \rangle$ is strictly m.d it is also m.d.

We say that a sequence $\langle a_n \rangle$ is <u>monotonic</u> if it is m.i or m.d (i.e., $a_1 \leq a_2 \leq a_3 \leq a_4 \leq \ldots\ldots$ or $a_1 \geq a_2 \geq a_3 \geq a_4 \geq \ldots\ldots$)

<u>Examples</u>:

(1) Consider the constant sequence $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, ..., i.e., the sequence $\langle a_n \rangle$ where a_n is equal to $\frac{1}{2}$ for all n. Then, $\langle a_n \rangle$ is both m.i and m.d.

(2) Consider the sequence $\langle (-1)^n \rangle$, i.e., -1, 1, -1, 1, ... This sequence is neither m.i. nor m.d.

(3) Consider the sequence given recursively by $a_{n+1} = \dfrac{a_n}{1 + a_n}$ for all $n \in N$. This sequence is given, <u>once the value of a_1 is given</u>. Then, by the recurrence equation, we get all the terms. Let a_1 have a value such that $a_1 > 0$. Then we see that for all

n, $a_n > 0$. Also, $a_n - a_{n+1} = a_n - \dfrac{a_n}{1+a_n} = \dfrac{a_n^{\,2}}{1+a_n} > 0$. So, $a_n > a_{n+1}$ for all n and hence $\langle a_n \rangle$ is m.d.

Question: What happens when $a_1 = 0$?

Definition 2: Suppose $\langle a_n \rangle$ is a sequence and $n_0 \in N$ and n_0 is a constant. Then:

(1) (a) If for all n such that $n \geq n_0$, $a_n \geq a_{n+1}$ (i.e., $a_{n_0} \geq a_{n_0+1} \geq a_{n_0+2} \geq ....$), we say that $\langle a_n \rangle$ is eventually monotonic decreasing.

(b) If for all n such that $n \geq n_0$, $a_n > a_{n+1}$ (i.e., $a_{n_0} > a_{n_0+1} > a_{n_0+2} > ....$) we say that $\langle a_n \rangle$ is eventually strictly monotonic decreasing.

(2) (a) If for all n such that $n \geq n_0$, $a_n \leq a_{n+1}$ (i.e., $a_{n_0} \leq a_{n_0+1} \leq a_{n_0+2} \leq ....$), we say that $\langle a_n \rangle$ is eventually monotonic increasing.

(b) If for all n such that $n \geq n_0$, $a_n < a_{n+1}$ (i.e., $a_{n_0} < a_{n_0+1} < a_{n_0+2} < ....$) we say that $\langle a_n \rangle$ is eventually strictly monotonic increasing.

Note: The 'note' given in definition 1 applies here too.

Example: Consider the sequence $\left\langle \dfrac{1}{2n-1} \right\rangle$, i.e., -1, 1, $\frac{1}{5}, \frac{1}{7}, \frac{1}{9}$, ...

Let $a_n = \dfrac{1}{2n-1}$. We see that for all n such that $n \geq 2$ (i.e., we can take $n_0 = 2$ here), $a_n > a_{n+1}$. So, $\left\langle \dfrac{1}{2n-1} \right\rangle$ is eventually strictly monotonic decreasing. It is also, eventually monotonic decreasing.

2.1.6. Bounded Sequences

Definition: Suppose $\langle a_n \rangle$ is a sequence. Then:

(1) If for all n, $a_n \leq k$ where k is a real number constant, we say that $\langle a_n \rangle$ is bounded above.

(2) If for all n, $a_n \geq k$ where k is a real number constant, we say that $\langle a_n \rangle$ is bounded below.

(3) If $\langle a_n \rangle$ is both bounded above and bounded below, we say that $\langle a_n \rangle$ is bounded.

We see that $\langle a_n \rangle$ is bounded means for all n, $k_1 \le a_n \le k_2$ where k_1, k_2 are real number constants.
We also have that, $\langle a_n \rangle$ is bounded if and only if $|a_n| \le k$, where k is a real number constant.

Examples:

(1) Consider the sequence $\langle (-1)^n \rangle$, i.e., -1, 1, -1, 1, …. For all n, $-1 \le (-1)^n \le 1$. Hence, $\langle (-1)^n \rangle$ is bounded. Also, for all n, $|(-1)^n| \le 1$.

(2) Consider the sequence $\langle n \rangle$, i.e., 1, 2, 3, 4, …. For all n, $n \ge 0$ (actually $n \ge 1$)
Therefore, $\langle n \rangle$ is bounded below.
However, $\langle n \rangle$ is not bounded above.
Hence $\langle n \rangle$ is not bounded.

(3) Consider the sequence $\langle -n \rangle$, i.e., -1, -2, -3, …… For all n, $-n \le 0$. Hence, $\langle -n \rangle$ is bounded above. However $\langle -n \rangle$ is not bounded below and hence it is not bounded.

(4) Let us reconsider example (3) of 2.1.5, i.e., the sequence $\langle a_n \rangle$ given by the recurrence equation $a_{n+1} = \dfrac{a_n}{1+a_n}$ and the value of a_1. Consider when $a_1 \ge 0$.
Then, for all n, $a_n \ge 0$ and $1 \ge \dfrac{a_n}{1+a_n} \ge 0$.
Let $K = \max\{a_1, 1\}$ (i.e., K is the greatest value in the set $\{a_1, 1\}$).
Then $0 \le a_n \le K$ for all n.
Therefore, $\langle a_n \rangle$ is bounded.

2.1.7. Relationship between monotonicity and boundedness

Theorem: Suppose $\langle a_n \rangle$ is monotonic (or eventually monotonic).

(1) If $\langle a_n \rangle$ is m.i (or eventually monotonic) and bounded above, then, $\langle a_n \rangle$ is convergent.

(2) If $\langle a_n \rangle$ is <u>m.i</u> (or <u>eventually monotonic</u>) and <u>not bounded above</u>, then, $\lim_{n\to\infty} a_n = \infty$.

(3) If $\langle a_n \rangle$ is <u>m.d</u> (or <u>eventually monotonic</u>) and <u>bounded below</u>, then, $\langle a_n \rangle$ is convergent.

(4) If $\langle a_n \rangle$ is <u>m.d</u> (or <u>eventually monotonic</u>) and <u>not bounded below</u>, then, $\lim_{n\to\infty} a_n = -\infty$.

<u>Note</u>:

(i) In (1), if for all n such that $n \geq n_o$, $a_n \leq k$ where $n_o \in N$ and $k \in R$, then $\lim_{n\to\infty} a_n \leq k$ (see theorem (4) in 2.1.4.)

(ii) In (3), if for all n such that $n \geq n_o$, $a_n \geq k$ where $n_o \in N$ and $k \in R$, then $\lim_{n\to\infty} a_n \geq k$ (see theorem (4) in 2.1.4.)

<u>Examples</u>:

(1) Consider the sequence $\langle a_n \rangle$ where for all n, $a_n = \dfrac{2n}{n+1}$.

$a_n = \dfrac{2(n+1)-2}{n+1} = 2 - \dfrac{2}{n+1}$.

As n increases, $\dfrac{2}{n+1}$ decreases and hence a_n increases. Therefore, $\langle a_n \rangle$ is m.i.

Also, for all n, $a_n = 2 - \dfrac{2}{n+1} \leq 2$. Therefore, $\langle a_n \rangle$ is bounded above.

Therefore, $\langle a_n \rangle$ is convergent.

In fact from the algebra of limits we have, $\lim_{n\to\infty}(2 - \dfrac{1}{n+1}) = 2 - 0 = 2$.

(2) Consider the sequence $\langle a_n \rangle$ where for all n, $a_n = n^2$. We have that, $\langle a_n \rangle$ is m.i and not bounded above. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty$.

(3) Consider the sequence $\langle a_n \rangle$ where for all n, $a_n = \dfrac{1}{n}$.

$\langle a_n \rangle$ is m.d and bounded below (for all n, $a_n \geq 0$).

Therefore $\langle a_n \rangle$ is convergent.

In fact, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{1}{n} = 0$.

(4) Consider the sequence $\langle a_n \rangle$ where for all n, $a_n = -n$.

Then, $\langle a_n \rangle$ is m.d but <u>not</u> bounded below.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-n) = -\infty$.

(5) Let us consider the sequence in example 3 of 2.1.5 and let us take $a_1 > 0$.
We saw that then, $\langle a_n \rangle$ is m.d.
In example 4 of 2.1.6, we saw that $\langle a_n \rangle$ is bounded above.
Therefore $\langle a_n \rangle$ is convergent.
So, $\lim_{n\to\infty} a_n = l$ for some $l \in R$. Let us find the value of l.

$a_{n+1} = \dfrac{a_n}{1+a_n}$ for all n.

As $n \to \infty$, $n+1 \to \infty$ and hence $\lim_{n\to\infty} a_{n+1} = l$.

Therefore, $l = \lim_{n\to\infty} \dfrac{a_n}{1+a_n} = \dfrac{l}{1+l}$.

i.e., $l + l^2 = l$.
i.e., $l^2 = 0$.
Therefore, $l = 0$.

*<u>Note</u>: For exercises/Further examples see Ref. 5: pages 387 to 393.

2.2 <u>Infinite Series</u>

2.2.1 Definition: Suppose $\langle a_n \rangle$ is a sequence. We form the sequence $\langle S_n \rangle$ where for $n \in N$, $S_n = a_1 + a_2 + ... + a_n = \sum_{i=1}^{n} a_i$.

We say that $\langle S_n \rangle$ is the sequence of partial sums and S_n is the sum of the first n terms of the sequence $\langle a_n \rangle$.

$S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$ etc.

We call the sequence $\langle S_n \rangle$ a series and it is denoted by $\sum_{n=1}^{\infty} a_n$.

S_1, S_2, S_3 etc are called the terms of the series.

Example: Consider the sequence $\langle a_n \rangle$ where $a_n = \frac{1}{2^{n-1}}$.

Consider the series $\sum_{n=1}^{\infty} a_n$, i.e., $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$.

This is a geometric series and we know that,

$$S_n = \sum_{i=1}^{n} \frac{1}{2^{i-1}} = 1 + \frac{1}{2} + ... + \frac{1}{2^{n-1}} = \frac{(1 - \frac{1}{2^n})}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}}.$$

2.2.2 Definition: Consider a series $\sum_{n=1}^{\infty} a_n$, i.e., the series $\langle S_n \rangle$ where $S_n = a_1 + a_2 + ... + a_n$. If $\langle S_n \rangle$ is convergent we say that the series $\sum_{n=1}^{\infty} a_n$ converges (or $\sum_{n=1}^{\infty} a_n$ is convergent).

In this case, $\lim_{n \to \infty} S_n = l$, for some $l \in R$.

If $\langle S_n \rangle$ is divergent (i.e., it is not convergent), we say that the series $\sum_{n=1}^{\infty} a_n$ diverges. (or $\sum_{n=1}^{\infty} a_n$ is divergent).

In the case $\sum_{n=1}^{\infty} a_n$ is convergent and $\lim_{n \to \infty} S_n = l$, we write $\sum_{n=1}^{\infty} a_n = l$.

Examples:

(1) Reconsider the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ (see 2.2.1)

We saw that, $S_n = \sum_{i=1}^{n} \frac{1}{2^{i-1}} = 1 + \frac{1}{2} + ... + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}$.

Now $\lim_{n\to\infty} S_n = 2$ (since $\lim_{n\to\infty} \frac{1}{2^{n-1}} = \lim_{n\to\infty} (\frac{1}{2})^{n-1} = 0$).

Therefore, $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is convergent and it converges to 2.

(2) Consider the series $\sum_{n=1}^{\infty} a_n$ where for any n, $a_n = \frac{1}{n(n+1)}$.

$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Let $S_n = \sum_{r=1}^{n} \frac{1}{r(r+1)}$ for any $n \in N$.

Then, $S_n = \sum_{r=1}^{n} (\frac{1}{r} - \frac{1}{r+1}) = \sum_{r=1}^{n} \frac{1}{r} - \sum_{r=1}^{n} \frac{1}{r+1}$

$$= (1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}) - (\frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n+1})$$

$$= 1 - \frac{1}{n+1}$$

(Note: We can also get this result in the following way:

$$\sum_{r=1}^{n} \frac{1}{r} - \sum_{r=1}^{n} \frac{1}{r+1} = \sum_{r=1}^{n} \frac{1}{r} - \sum_{r=2}^{n+1} \frac{1}{r} = 1 - \frac{1}{n+1})$$

Therefore, $\lim_{n\to\infty} S_n = 1$ (since, $\lim_{n\to\infty} \frac{1}{n+1} = 0$)

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and it converges to 1.

(3) Consider the series $\sum_{n=1}^{\infty} (-1)^{n+1}$, i.e., $\sum_{n=1}^{\infty} a_n$ where for any n, $a_n = (-1)^{n+1}$.

Let $S_n = a_1 + a_2 + ... + a_n$, for any n.

Then, $S_n = 1 - 1 + 1 - ....$, to n terms,

So, $S_n = 1 - 1 + 1 - 1 + ... + 1 - 1 = 0$ when n is even, and

$S_n = 1 - 1 + 1 - 1 + ... + 1 - 1 + 1 = 1$ when n is odd.

Therefore, $\langle S_n \rangle$ is divergent.

Therefore, $\sum_{n=1}^{\infty} a_n$ is divergent, i.e., $\sum_{n=1}^{\infty} (-1)^{n+1}$ is divergent.

(4) Consider the series $\sum_{n=1}^{\infty} a_n$ where for any n, $a_n = n^2$.

The sequence of partial sums $\langle S_n \rangle$ is given by $S_n = 1^2 + 2^2 + ... + n^2$.

For any n, $S_n = 1^2 + 2^2 + ... + n^2 \geq n^2$.

i.e., for any n, $S_n \geq n^2$ and $\lim_{n\to\infty} n^2 = \infty$.

Therefore, $\lim_{n\to\infty} S_n = \infty$.

Therefore, $\langle S_n \rangle$ is divergent.

Therefore, $\sum_{n=1}^{\infty} n^2$ is divergent.

(5) Consider the series $\sum_{n=1}^{\infty} -n^2$.

The sequence of partial sums $\langle S_n \rangle$ is given by, $S_n = - 1^2 - 2^2 - \ldots - n^2$.

Therefore for any n, $S_n \leq -n^2$ and $\lim_{n\to\infty} - n^2 = -\infty$.

Therefore $\lim_{n\to\infty} S_n = -\infty$.

Therefore $\langle S_n \rangle$ is divergent.

Therefore $\sum_{n=1}^{\infty} -n^2$ is divergent.

Note: A series $\sum_{n=1}^{\infty} a_n$ could be divergent in any one of the following three ways:

Let $\langle S_n \rangle$ be the sequence of partial sums.

(i) $\lim_{n\to\infty} S_n = \infty$

(ii) $\lim_{n\to\infty} S_n = -\infty$

(iii) Neither $\lim_{n\to\infty} S_n = \infty$ nor $\lim_{n\to\infty} S_n = -\infty$ nor $\lim_{n\to\infty} S_n = l$ for some $l \in R$.

(In this case we say that the sequence $\langle S_n \rangle$ is oscillatory)

Possibility (i) is found in Example (4).
Possibility (ii) is found in Example (5).
Possibility (iii) is found in Example (3).

2.2.3 Fundamental Facts about Infinite Series

Theorem 1: Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. Let $c, \lambda_1, \lambda_2 \in R$ be constants.

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent.

Then:

(1) $\sum_{n=1}^{\infty} ca_n$ is convergent and $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$.

(Note: For $n \in N$, let $T_n = \sum_{r=1}^{n} ca_r$ and $S_n = \sum_{r=1}^{n} a_r$ Then, for any $n \in N$, $T_n = cS_n$ and if $\lim_{n\to\infty} S_n = l$, then, $\lim_{n\to\infty} T_n = cl$).

(2) $\sum_{n=1}^{\infty}(a_n + b_n)$ is convergent and $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

(Note: For $n \in N$, let $S_n = \sum_{r=1}^{n} a_r$ and $T_n = \sum_{r=1}^{n} b_r$ and $U_n = \sum_{r=1}^{n}(a_r + b_r)$. Then, for any $n \in N$, $U_n = S_n + T_n$, and if $\lim_{n\to\infty} S_n = l_1$ and $\lim_{n\to\infty} T_n = l_2$, then, $\lim_{n\to\infty} U_n = l_1 + l_2$).

(3) $\sum_{n=1}^{\infty}(\lambda_1 a_n + \lambda_2 b_n)$ is convergent and $\sum_{n=1}^{\infty}(\lambda_1 a_n + \lambda_2 b_n) = \lambda_1 \sum_{n=1}^{\infty} a_n + \lambda_2 \sum_{n=1}^{\infty} b_n$

(Note: This follows from (1) and (2). Also, when $\lambda_1 = 1$ and $\lambda_2 = -1$, we get, $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n$ - $\sum_{n=1}^{\infty} b_n$).

Theorem 2: Suppose $\langle a_n \rangle$ is a sequence of non-negative terms and $\langle S_n \rangle$ is a sequence of partial sums.

Then, $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\langle S_n \rangle$ is bounded above (i.e., for all n, $S_n \le k$ for some constant $k \in R$).

(Note: See the Theorem in 2.1.7)

Examples:

(1) Suppose $\langle t_n \rangle$ is a sequence such that for all n, $t_n \in \{0,1,2,3,4,...9\}$ and $\langle a_n \rangle$ is the sequence where for any n, $a_n = \frac{t_n}{10^n}$.

Now consider the series $\sum_{n=1}^{\infty} a_n$.

(Note: This is actually the infinite decimal, $0.t_1t_2t_3.....$)

Let $\langle S_n \rangle$ be the sequence of partial sums.

Then for any n, $S_n = \frac{t_1}{10} + \frac{t_2}{10^2} + ... + \frac{t_n}{10^n} \le \frac{9}{10} + \frac{9}{10^2} + ... + \frac{9}{10^n}$.

$$\frac{9}{10}+\frac{9}{10^2}+.....+\frac{9}{10^n}=\frac{9}{10}(1+\frac{1}{10}+......+\frac{1}{10^{n-1}})=\frac{9}{10}(\frac{1-\frac{1}{10^n}}{1-\frac{1}{10}})=1-\frac{1}{10^n}\leq 1.$$

Therefore, for all n, $S_n \leq 1$; i.e., $\langle S_n \rangle$ is bounded above.

Also, $\langle a_n \rangle$ is a sequence of non-negative terms and hence $\langle S_n \rangle$ is m.i.

Therefore, $\sum_{n=1}^{\infty} a_n$ is convergent, i.e., $\sum_{n=1}^{\infty} \frac{t_n}{10^n}$ is convergent.

(Note: The infinite decimal $0.t_1t_2t_3.....$ is actually $\sum_{n=1}^{\infty} \frac{t_n}{10^n}$)

(2) Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{n}$. Then $\langle a_n \rangle$ is a sequence of non-negative terms. Let $\langle S_n \rangle$ be the sequence of partial sums.

Let $n \in N$.

Then
$$\begin{aligned} S_{2^{n+1}-1} &= 1+(\frac{1}{2}+\frac{1}{3})+(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7})+....+(\frac{1}{2^n}+\frac{1}{2^n+1}+....+\frac{1}{2^{n+1}-1}) \\ &\geq 1+(2\times\frac{1}{4})+(4\times\frac{1}{8})+.....+(2^n\times\frac{1}{2^{n+1}}) \\ &= 1+\underbrace{\frac{1}{2}+\frac{1}{2}+....+\frac{1}{2}}_{\text{to } n \text{ terms}} \\ &= 1+\frac{n}{2} \end{aligned}$$

Therefore, $S_{2^{n+1}-1} \geq 1+\frac{n}{2}$ and $\lim_{n\to\infty}(1+\frac{n}{2}) = \infty$

Therefore, for any $k \in R$, $S_{2^{n+1}-1} \geq k$ for some n.

Therefore, $\langle S_n \rangle$ is not bounded above.

Also, $\langle S_n \rangle$ is m.i.

Therefore, $\langle S_n \rangle$ is divergent.

Therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Theorem 3 (Divergence Test):

If, not $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Note:

(1) By logic, we have that this is the same as saying 'If $\sum_{n=1}^{\infty} a_n$ is convergent, then

$\underline{\lim_{n\to\infty} a_n = 0}$'. (However, <u>we cannot say that</u>, <u>if</u> $\lim_{n\to\infty} a_n = 0$ <u>then</u> $\sum_{n=1}^{\infty} a_n$ is convergent. For instance, $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

(2) Consider the situation when, not $\lim_{n\to\infty} a_n = 0$.

This can be so in any one of the following four ways.

(i) $\lim_{n\to\infty} a_n = l$ for some l such that $l \neq 0$.

(ii) $\lim_{n\to\infty} a_n = \infty$

(iii) $\lim_{n\to\infty} a_n = -\infty$

(iv) Neither $\langle a_n \rangle$ is convergent, nor $\lim_{n\to\infty} a_n = \infty$, nor $\lim_{n\to\infty} a_n = -\infty$.

<u>Examples</u>:

(1) Consider the general <u>geometric series</u> $\sum_{n=1}^{\infty} ar^{n-1}$ (i.e., $a + ar + ar^2 + \ldots..$)

<u>When $r > 1$</u>:

$\lim_{n\to\infty} r^{n-1} = \infty$ (As $n \to \infty, n-1 \to \infty$ and $\lim_{n\to\infty} r^n = \infty$ (see 2.1.4), we have as $n \to \infty$, $r^{n-1} \to \infty$)

Therefore, when $a > 0$, $\lim_{n\to\infty} ar^{n-1} = \infty$

and when $a < 0$, $\lim_{n\to\infty} ar^{n-1} = -\infty$.

Therefore, not, $\lim_{n\to\infty} ar^{n-1} = 0$.

Therefore, when $r > 1$, $\sum_{n=1}^{\infty} ar^{n-1}$ is divergent.

<u>When $r = 1$</u>:

$ar^{n-1} = a \neq 0$. Therefore, $\lim_{n\to\infty} ar^{n-1} = a \neq 0$.

Therefore, not, $\lim_{n\to\infty} ar^{n-1} = 0$.

Therefore, when $r = 1$, $\sum_{n=1}^{\infty} ar^{n-1}$ is divergent.

<u>When $r = -1$</u>:

$$ar^{n-1} = a \times (-1)^{n-1} = \begin{cases} a & \text{when } n \text{ is odd} \\ -a & \text{when } n \text{ is even} \end{cases}$$

Therefore, neither $\langle ar^{n-1} \rangle$ is convergent, nor $\lim_{n\to\infty} ar^{n-1} = \infty$, nor $\lim_{n\to\infty} ar^{n-1} = -\infty$.

Therefore, not, $\lim_{n\to\infty} ar^{n-1} = 0$.

$\boxed{\text{Therefore, when } r = -1,\ \sum_{n=1}^{\infty} ar^{n-1} \text{ is divergent.}}$

<u>When $r < -1$</u>:

$\left|ar^{n-1}\right| = |a|\left|r^{n-1}\right| = |a||r|^{n-1} \geq |a|$ (as $|r| > 1$) and $|a| > 0$.

Therefore, not $\lim\limits_{n\to\infty}\left|ar^{n-1}\right| = 0$ (see 2.1.4, Theorem 4)

Therefore, not $\lim\limits_{n\to\infty} ar^{n-1} = 0$ (see the last result in 2.1.4).

$\boxed{\text{Therefore, when } r < -1,\ \sum_{n=1}^{\infty} ar^{n-1} \text{ is divergent.}}$

This brings us to the final possibility.

<u>When $-1 < r < 1$</u>:

$\lim\limits_{n\to\infty} r^n = 0$ (see 2.1.4)

Therefore, $\lim\limits_{n\to\infty} ar^{n-1} = 0$.

However, <u>this does not ensure that</u> $\sum\limits_{n=1}^{\infty} ar^{n-1}$ is convergent.

Let $\langle S_n \rangle$ be the sequence of partial sums.

$S_n = \dfrac{a(1-r^n)}{1-r} = \dfrac{a}{1-r} - \dfrac{a}{1-r} \times r^n \quad \to \dfrac{a}{1-r} - 0$ as $n \to \infty$, since $r^n \to 0$ as $n \to \infty$.

Therefore $\lim\limits_{n\to\infty} S_n = \dfrac{a}{1-r}$.

$\boxed{\text{Therefore, when} -1 < r < 1,\ \sum_{n=1}^{\infty} ar^{n-1} \text{ is convergent and } \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.}$

(2) Consider the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{\lambda}}$ where $\lambda \in R$ and is a constant. We saw that when $\lambda = 1$, $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{\lambda}}$ is divergent. (i.e., $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ is divergent).

Let us consider the situation for the other values of λ.

<u>When $\lambda < 0$</u>:

$\lambda = -k$ where $k = -\lambda > 0$.

$\dfrac{1}{n^{\lambda}} = n^k$ an $\lim\limits_{n\to\infty} n^k = \infty$. ($\lim\limits_{n\to\infty} n^k = \infty$ when $k > 0$ is a standard limit).

Therefore, $\lim\limits_{n\to\infty} \dfrac{1}{n^{\lambda}} = \infty$ and hence by the theorem, $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{\lambda}}$ is divergent.

Therefore, $\sum_{n=1}^{\infty}\frac{1}{n^{\lambda}}$ is divergent when λ < 0.

When λ = 0:

$\frac{1}{n^{\lambda}}=1$ and hence $\lim_{n\to\infty}\frac{1}{n^{\lambda}}=1$.

Therefore, not $\lim_{n\to\infty}\frac{1}{n^{\lambda}}=0$.

Therefore, $\sum_{n=1}^{\infty}\frac{1}{n^{\lambda}}$ is divergent when λ = 0.

When 0 < λ < 1:

We will postpone the consideration of this case. We will consider it after the next theorem (i.e., Theorem 4).

When λ > 1:

Although $\lim_{n\to\infty}\frac{1}{n^{\lambda}}=0$, this does not ensure that $\sum_{n=1}^{\infty}\frac{1}{n^{\lambda}}$ is convergent. However, it is in fact convergent and we shall now show this:

Let $\langle S_n \rangle$ be the sequence of partial sums.

Let $n \in N$.

$$S_{2^{n+1}-1}=1+(\frac{1}{2^{\lambda}}+\frac{1}{3^{\lambda}})+(\frac{1}{4^{\lambda}}+\frac{1}{5^{\lambda}}+\frac{1}{6^{\lambda}}+\frac{1}{7^{\lambda}})+....+(\frac{1}{(2^n)^{\lambda}}+\frac{1}{(2^n+1)^{\lambda}}+...+\frac{1}{(2^{n+1}-1)^{\lambda}})$$

$$\le 1+(\frac{1}{2^{\lambda}}\times 2)+(\frac{1}{4^{\lambda}}\times 4)+.....+(\frac{1}{(2^n)^{\lambda}}\times 2^n)$$

$$=1+\frac{1}{2^{\lambda-1}}+\frac{1}{(2^{\lambda-1})^2}+....+\frac{1}{(2^{\lambda-1})^n}$$

$$=1+r+r^2+....+r^n \text{ where } r=\frac{1}{2^{\lambda-1}} \text{ and } \underline{0<r<1}. \text{ (as } \lambda-1>0)$$

$$=\frac{1-r^{n+1}}{1-r}$$

$$=\frac{1}{1-r}-\frac{r^{n+1}}{1-r}\le\frac{1}{1-r}$$

Therefore, $\underline{S_{2^{n+1}-1}\le\frac{1}{1-r}}$ where $\underline{r=\frac{1}{2^{\lambda-1}}}$.

Let $m \in N$. Then, there is $n \in N$ such that $2^{n+1}-1>m$ (since, $\lim_{n\to\infty}(2^{n+1}-1)=\infty$). Since the terms of the series are non-negative and $2^{n+1}-1>m$, $S_m \le S_{2^{n+1}-1}\le\frac{1}{1-r}$.

Therefore for any $m \in N$, $S_m \le \frac{1}{1-r}$.

Therefore, $\langle S_n \rangle$ is m.i and bounded above.

Therefore, $\langle S_n \rangle$ is convergent.

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^\lambda}$ is convergent when $\lambda > 1$.

In the next two theorems we will consider only series of positive terms (i.e., series $\sum_{n=1}^{\infty} a_n$ where for all $n \in N, a_n > 0$).

Theorem 4 (Comparison Test):

Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be series of positive terms and let $n_0 \in N$, be a constant for which, $a_n \le b_n$ for all n such that $n \ge n_0$.

Then:

If $\sum_{n=1}^{\infty} b_n$ is convergent, then, $\sum_{n=1}^{\infty} a_n$ is also convergent.

(Note: By logic, this is the same as saying, 'If $\sum_{n=1}^{\infty} a_n$ is divergent, then, $\sum_{n=1}^{\infty} b_n$ is divergent')

Example:

We said in Example 2 of Theorem 3 that we will be considering here the series $\sum_{n=1}^{\infty} \frac{1}{n^\lambda}$ when $0 < \lambda < 1$.

Let $0 < \lambda < 1$.

Then for any $n \in N$, $\frac{1}{n} \le \frac{1}{n^\lambda}$ (since, $n^\lambda \le n$ when $0 < \lambda < 1$)

But $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^\lambda}$ is divergent; i.e., $\sum_{n=1}^{\infty} \frac{1}{n^\lambda}$ is divergent when $0 < \lambda < 1$.

So, we have from this and what we had in Example 2 of Theorem 3, that,

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^\lambda} \text{ is convergent if and only if } \lambda > 1}$$

Theorem 5 (Limit Comparison Test):

Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be series of positive terms and $\lim_{n\to\infty}(\frac{a_n}{b_n}) = l$ for some $l \in R$.

(Note: By Theorem 4 of 2.1.4, we have that $l \geq 0$ since for all $n \in N$, $\frac{a_n}{b_n} > 0$).

Then:

If $l > 0$, $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\sum_{n=1}^{\infty} b_n$ is convergent, and if $l = 0$, when $\sum_{n=1}^{\infty} b_n$ is convergent, $\sum_{n=1}^{\infty} a_n$ is also convergent.

*Remark: For any given series $\sum_{n=1}^{\infty} x_n$ if we change the values of a finite number of terms (for example consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{n}$. Now consider the series $\sum_{n=1}^{\infty} b_n$ where $b_1 = 0$ and $b_5 = 0$ and $b_{12} = \frac{1}{2^{12}}$ and $b_{100} = \frac{1}{2^{100}}$ and for the other values of n, $b_n = a_n$. Here $\sum_{n=1}^{\infty} b_n$ has been obtained by changing the values of a finite number of terms of $\sum_{n=1}^{\infty} a_n$, i.e., $\sum_{n=1}^{\infty} \frac{1}{n}$), then, the convergence or divergence of the new series is the same as that of the original series.

Let $\langle S_n \rangle$ be the sequence of partial sums of the original series and let $\langle T_n \rangle$ be the sequence of partial sums of the new series. Then, $T_n = S_n + k$ for all n such that $n \geq n_0$, where $k \in R$ and $n_0 \in N$ are constants. On the other hand, if we delete a finite number of terms (for example consider a series $\sum_{n=1}^{\infty} a_n$ and we form a series $\sum_{n=1}^{\infty} b_n$ by deleting a_1, a_2, a_3 and a_{14}), with the notation used above we have, $T_n = S_{n+n_0} + k$ for all n such that $n \geq n_1$, where $k \in R$ and $n_0, n_1 \in N$ are constants.

If both these changes are made, we get T_n in the above form.

In all these, the convergence or divergence of the new series is the same as that of the original series.

* So, although we insisted in Theorems 4 and 5, that for all n, $a_n > 0$ and $b_n > 0$, the conclusions of these two theorems still remains true if, for all n such that $n \geq n_0$, $a_n > 0$ and $b_n > 0$ where $n_0 \in N$ is a constant.

Do the exercises on Page 276 of Ref 1.

Alternating series: A series whose terms are alternately positive and negative is said to be an alternating series. An alternating series is of the form $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ where for all n, $a_n > 0$.

Theorem 6 (Alternating Series Theorem):

Let $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ be an alternating series and $\langle a_n \rangle$ be m.d and $\lim_{n\to\infty} a_n = 0$. Then, $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent.

Example:

Consider the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-....$, i.e., the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ where $a_n = \frac{1}{n}$. Then, $\langle a_n \rangle$ is m.d and $\lim_{n\to\infty} a_n = 0$. Therefore the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is convergent.

Absolute Convergence: A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty}|a_n|$ is convergent.

Theorem 7: Suppose $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Then, $\sum_{n=1}^{\infty} a_n$ is convergent.

Definition: A series is said to be conditionally convergent if it is convergent but it is not absolutely convergent.

Examples:

(1) Consider the alternating serires $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$. $\left|(-1)^{n+1}\frac{1}{n}\right| = \frac{1}{n}$ and so $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n}\right|$ is $\sum_{n=1}^{\infty}\frac{1}{n}$ which is divergent. Therefore $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is conditionally convergent. (See the above example)

(2) The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ is absolutely convergent.

The Ratio Test and the Root Test:

Theorem 8 (Ratio Test): Consider a series $\sum_{n=1}^{\infty} a_n$ where for all n, $a_n \neq 0$ (or at least for all $n \geq n_0$, $a_n \neq 0$ where $n_0 \in N$ is a constant). Then:

(1) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = l$ and $l < 1$, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(2) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = l$ and $l > 1$, $\sum_{n=1}^{\infty} a_n$ is divergent.

(3) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, $\sum_{n=1}^{\infty} a_n$ is divergent.

Theorem 9 (Root Test): Consider a series $\sum_{n=1}^{\infty} a_n$.

(1) If $\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = l$ and $l < 1$, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(2) If $\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = l$ and $l > 1$, $\sum_{n=1}^{\infty} a_n$ is divergent.

(3) If $\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \infty$, $\sum_{n=1}^{\infty} a_n$ is divergent.

Examples:

(1) Consider $\sum_{n=0}^{\infty} a_n$ where $a_n = \frac{1}{n!}$. The convergence or divergence, as the case may be, is the same as that of $\sum_{n=1}^{\infty} a_n$ and we will consider this series (see * remark appearing just after Theorem 5).

Then, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0$ as $n \to \infty$. (i.e., $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$). Since $0 < 1$, by the Ratio Test, $\sum_{n=1}^{\infty}\frac{1}{n!}$ is convergent and hence, $\sum_{n=0}^{\infty}\frac{1}{n!}$ is convergent.

(Note: (i) It is absolutely convergent, but since the terms are positive, it is more sensible to say that it is convergent.

(ii) $\sum_{n=0}^{\infty}\frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ...$, while $\sum_{n=1}^{\infty}\frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + ...$)

* Note: $\sum_{n=0}^{\infty}\frac{1}{n!}$ is denoted by e and in logarithms, $\log_e x$ is called the natural logarithm of x and it is written as $\ln x$.

In this context, we also mention another standard limit (see 2.1.4), namely $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

We also considered this series in 2.1.4 as an example for Theorem 4 and we showed there that $e \geq 2.5$.

(2) Consider the series $\sum_{n=1}^{\infty} a_n$, where $a_n = \frac{n^n}{n!}$.

Then, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \frac{(n+1)^n}{n^n} = (1+\frac{1}{n})^n$.

Now, we know that, $\lim_{n\to\infty}(1+\frac{1}{n})^n = e \geq 2.5$.

Therefore, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = e > 1$.

Therefore, $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ is divergent.

Note: This can be more easily proved using the Divergence Test (Theorem 3) as $n^n = \underbrace{n \times n \times .....\times n}_{n\text{ factors}} \geq 1\times 2\times .....\times n = n!$ for all n and hence $\frac{n^n}{n!} \geq 1$ for all n.

(3) The sequence $\langle a_n \rangle$ is given by the recursion formula, $a_{n+1} = \frac{n^2}{n+1}\times a_n$ for any $n \in N$, and $a_1 = -1$.

Then,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n^2}{n+1} = \lim_{n\to\infty} n\times(\frac{1}{1+\frac{1}{n}}) = \infty$$

Therefore, $\sum_{n=1}^{\infty} a_n$ is divergent.

(4) Consider the series $\sum_{n=1}^{\infty}\frac{n}{2^n}$, i.e., the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{n}{2^n}$.

Then, $|a_n|^{\frac{1}{n}} = \left(\frac{n}{2^n}\right)^{\frac{1}{n}} = \frac{n^{\frac{1}{n}}}{2}$. Since, $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$ (see 2.1.4),

$\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{2} < 1$.

Therefore, $\sum_{n=1}^{\infty} \frac{n}{2^n}$ is convergent.

Exercise: Use the Ratio Test to show that this series is convergent.

(5) Consider the series $\sum_{n=1}^{\infty} \frac{n^n + 1}{2^n - 1}$.

$\left|\frac{n^n + 1}{2^n - 1}\right| = \frac{n^n + 1}{2^n - 1} \geq \frac{n^n}{2^n} = \left(\frac{n}{2}\right)^n$ for all $n \in N$.

Therefore, $\left|\frac{n^n + 1}{2^n - 1}\right|^{\frac{1}{n}} \geq \left(\left(\frac{n}{2}\right)^n\right)^{\frac{1}{n}} = \frac{n}{2} \to \infty$ as $n \to \infty$.

Therefore, $\lim\limits_{n\to\infty} \left|\frac{n^n + 1}{2^n - 1}\right|^{\frac{1}{n}} = \infty$.

Therefore, $\sum_{n=1}^{\infty} \frac{n^n + 1}{2^n - 1}$ is divergent.

Use some of the steps in the above proof and with the use of the Comparison Test (Theorem 4), get the above result by using the Ratio Test.

(6) Consider the series $\sum_{n=1}^{\infty} \frac{e^n}{n+1}$. For all $n \in N$, $\frac{e^n}{2n} \leq \frac{e^n}{n+1}$ (since $n + 1 \leq 2n$).

$\left|\frac{e^n}{2n}\right|^{\frac{1}{n}} = \frac{e}{2^{\frac{1}{n}} \times n^{\frac{1}{n}}} \to e$ as $n \to \infty$ (since $\lim\limits_{n\to\infty} 2^{\frac{1}{n}} = \lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$)

Therefore, $\lim\limits_{n\to\infty} \left|\frac{e^n}{2n}\right|^{\frac{1}{n}} = e > 1$ (we know that $e \geq 2.5$)

Therefore, $\sum_{n=1}^{\infty} \frac{e^n}{2n}$ is divergent.

Therefore, by the Comparison Test, $\sum_{n=1}^{\infty} \frac{e^n}{n+1}$ is divergent.

In connection with determining whether a series is convergent or whether it is divergent, we may need a few more standard limits.

We give these standard limits without going into much detail:

(1) If $\lim_{n\to\infty} x_n = c$ and $\lim_{x\to c} f(x) = l$, then, $\lim_{n\to\infty} f(x_n) = l$.

For example, $\lim_{x\to 0}\frac{\sin x}{x} = 1$ and $\lim_{n\to\infty}\frac{1}{n} = 0$.

Therefore, $\lim_{n\to\infty}\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right) = 1$.

(2) Suppose $\lim_{n\to\infty} x_n = \infty$. Then:

(i) If $\lim_{x\to\infty} f(x) = \infty$, then, $\lim_{n\to\infty} f(x_n) = \infty$.

(ii) If $\lim_{x\to\infty} f(x) = -\infty$, then, $\lim_{n\to\infty} f(x_n) = -\infty$.

(3) Suppose $r, s \in R$ and $r, s > 0$ and r, s are constants.

Then, $\lim_{n\to\infty} n^r (\ln n)^{-s} = \infty$ (($\ln n)^{-s}$ is defined for all n such that $n \geq 2$) and,

$\lim_{n\to\infty} n^{-r} (\ln n)^{s} = 0$.

Exercise: Show that $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{n}{(\ln n)^5}$ for all n such that $n \geq 2$ (see * remark, just after Theorem 5), is divergent.

*Exercise: Consider $\sum_{n=1}^{\infty} b_n$ where $b_n = \frac{1}{\ln n}$ for all $n \geq 2$. Consider $\langle a_n \rangle$ where $a_n = \frac{1}{n^{\frac{1}{2}}}$. Show that $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$ and using Theorem 5 show that $\sum_{n=1}^{\infty} b_n$ is divergent. Deduce that $\sum_{n=1}^{\infty}\frac{1}{\ln(n+1)}$ is divergent (See * note)

*Note: We take this occasion to belatedly state a result on sequences.

Result on Sequences: Suppose $\langle a_n \rangle$ is a sequence and k is an integer constant. Suppose $l \in R$. Then:

(i) $\lim_{n\to\infty} a_n = l$ if and only if $\lim_{n\to\infty} a_{n+k} = l$

(ii) $\lim_{n\to\infty} a_n = \infty$ if and only if $\lim_{n\to\infty} a_{n+k} = \infty$

(iii) $\lim_{n\to\infty} a_n = -\infty$ if and only if $\lim_{n\to\infty} a_{n+k} = -\infty$.

Note: When $k > 0$, $\langle a_{n+k} \rangle$ is the sequence $\langle b_n \rangle$ where for any n, $b_n = a_{n+k}$. (i.e., $b_1 = a_{k+1}$, $b_2 = a_{k+2}$ etc). For instance, when $k = 5$, $b_1 = a_6$, $b_2 = a_7$, $b_3 = a_8$ etc).

When $k < 0$, $\langle a_{n+k} \rangle$ is a sequence $\langle b_n \rangle$ where $b_n = a_{n+k}$ when $n \geq -k+1$ (i.e., $n + k \geq 1$), and when $1 \leq n \leq -k$, b_n could take whatever value we give them.

For example, $\langle a_{n-5} \rangle$ is a sequence $\langle b_n \rangle$ where for $n \geq 6$, $b_n = a_{n-5}$ (i.e., $b_6 = a_1$, $b_7 = a_2$, $b_8 = a_3$ etc) and b_1, b_2, b_3, b_4 and b_5 could take whatever value we give them.

So, in the previous exercise, if $\langle c_n \rangle$ is the sequence where $c_n = \dfrac{1}{\ln(n+1)}$ and $\langle T_n \rangle$ is the sequence of partial sums of the sequence (i.e., $T_1 = c_1$, $T_2 = c_1 + c_2$, $T_3 = c_1 + c_2 + c_3$ etc) and $\langle S_n \rangle$ is the sequence of partial sums of the sequence $\langle b_n \rangle$,

then, we have: $T_n = \dfrac{1}{\ln 2} + \dfrac{1}{\ln 3} + ... + \dfrac{1}{\ln(n+1)}$ and $S_n = b_1 + \dfrac{1}{\ln 2} + \dfrac{1}{\ln 3} + ... + \dfrac{1}{\ln n}$ and for any n such that $n \geq 2$, $T_n = S_{n+1} - b_1$, i.e., $T_n = S_{n+1}$ + constant.

From this we get, $\langle T_n \rangle$ is convergent if and only if $\langle S_n \rangle$ is convergent. Since we showed that $\langle S_n \rangle$ is divergent, we get that $\langle T_n \rangle$ is divergent, i.e., $\sum_{n=1}^{\infty} \dfrac{1}{\ln(n+1)}$ is divergent.

2.3 Power Series

2.3.1 Fundamental facts about Power Series

Definition:

Consider a series $\sum_{n=0}^{\infty} a_n (x-c)^n$ (i.e., $a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots\ldots$) where $\langle a_n \rangle$ is a sequence with first term a_0 and $x, c \in R$ and c is a constant while x is a variable. We say that, this is a power series about c. An important special case is when $c = 0$ where we have the power series about 0 which is $\sum_{n=0}^{\infty} a_n x^n$.

From a given context it is understood the number c about which it is a power series and we refer to any one of these series as just a power series.

Example: Consider the series $1 + x + x^2 + x^3 + \ldots\ldots$ This is the power series $\sum_{n=0}^{\infty} a_n x^n$ where $a_n = 1$ for all $n \in N \cup \{0\}$. It is also a geometric series and it is convergent only when $|x| < 1$ and $1 + x + x^2 + x^3 + \ldots. = \frac{1}{1-x}$ when $|x| < 1$, i.e., $-1 < x < 1$.

Theorem 1: Consider a power series $\sum_{n=0}^{\infty} a_n (x-c)^n$. Then, one and only one of the following possibilities occur:

(1) It converges for all values of x in R.
(2) It converges only when $x = c$.
(3) There is a real number R_1 , such that the power series converges for all $x \in R$ such that $c - R_1 < x < c + R_1$ (i.e., $|x - c| < R_1$) and diverges when $x > c + R_1$ and when $x < c - R_1$ (i.e., when $|x - c| > R_1$)

(Note: It may or may not converge when $x = c + R_1$. This is so also when $x = c - R_1$.)

The set of all values for which a power series converges is called its interval of convergence

Also: We say that the radius of convergence of the power series is
(i) ∞ in the case of (1).
(ii) 0, in the case of (2).
(iii) R_1 in the case of (3).

Examples:

(1) Consider the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. By application of the Ratio Test we get that the radius of convergence is ∞ (i.e., it converges for all $x \in R$)

(2) Consider the power series $\sum_{n=0}^{\infty} (n+1)^n (x-1)^n$. By applying the Root Test, we get that the radius of convergence is 0 and the series converges only when $x = 1$.

(3) Consider the power series $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ (i.e., $1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + ...$). By the Ratio Test we get that the radius of convergence is 2. It does not converge when $x = 2$ and when $x = -2$. Therefore, the interval of convergence is (-2, 2), i.e., the set of all $x \in R$ such that $-2 < x < 2$.

(4) Consider the power series $1 + x + \frac{x^2}{2} + \frac{x^3}{3} + ...$, i.e., $\sum_{n=0}^{\infty} a_n x^n$ where $a_n = \frac{1}{n}$ for all $n \geq 1$ and $a_0 = 1$.

In order to find the radius of convergence and the interval of convergence, it is sufficient to consider the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$. Then, by the Ratio Test we get that the radius of convergence is 1 and when $x = 1$ the series diverges (since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent) and when $x = -1$ the series converges by the Alternating Series Theorem (Theorem 6). (Note: Although in an alternating series the first term is positive, here the first term is negative. But this does not matter since, if we have a series $\sum_{n=1}^{\infty} (-1)^n a_n$ with $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ being an alternating series, taking $\langle S_n \rangle$ as the sequence of partial sums of the series $\sum_{n=1}^{\infty} (-1)^n a_n$ and $\langle T_n \rangle$ as the sequence of partial sums of the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, then, $T_n = - S_n$ and so both converge or diverge together.)

Therefore, the interval of convergence is [-1, 1), i.e., the set of all $x \in R$ such that $-1 \leq x < 1$.

Theorem 2 (Differentiation of Power Series)

Consider a power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ (i.e, $a_0 + a_1(x-c) + a_2(x-c)^2 + ...$) and suppose that the radius of convergence is R_1 for some R_1 such that $R_1 > 0$. Let I denote the interval of convergence. Let f be the function given by, $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, $x \in I$. Then, for any $x \in R$ such that, $c - R_1 < x < c + R_1$,

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + ...$$

f is differentiable at x when $|x-c| < R_1$ (i.e., $c - R_1 < x < c + R_1$) and $f'(x)$ is equal to the series $\sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$ (or $\sum_{n=0}^{\infty} (n+1) a_{n+1} (x-c)^n$) and this series also has radius of convergence R_1.

If the radius of convergence of $\sum_{n=0}^{\infty} a_n (x-c)^n$ is ∞, a similar result holds but in this case f is differentiable at x for all $x \in R$, and the series $\sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$ which is equal to $f'(x)$ also has ∞ as its radius of convergence.

*Note:

(1) The differential coefficient of $a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + ...$ is $a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + ...$ which is the series got by differentiating the terms of the given series.

*(2) We can repeatedly apply this theorem to get: For any $k \in N$,

$$f^k(x) = \sum_{n=k}^{\infty} n(n-1)....(n-k+1) a_n (x-c)^{n-k}.$$

(i.e., $f^k(x) = k! a_k + (2 \times 3 \times ... \times (k+1)) a_{k+1} (x-c) + (3 \times 4 \times ... \times (k+2)) a_{k+2} (x-c)^2 + ...$,

i.e., $f^k(x) = \frac{d^k}{dx^k}(a_k (x-c)^k) + \frac{d^k}{dx^k}(a_{k+1}(x-c)^{k+1}) + \frac{d^k}{dx^k}(a_{k+2}(x-c)^{k+2}) + ...$)

where x takes the values as it was for $f'(x)$.

*(3) All this applies when the radius of convergence is ∞ and here x takes all values in R, for $f'(x)$ and for $f^k(x)$ where $k \in N$ and all the series mentioned have radius of convergence ∞.

*Exercise/Result: Show that for any $k \in N$, $a_k = \frac{f^k(c)}{k!}$.

Theorem 3: Let $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, $c - R_1 < x < c + R_1$ where $R_1 > 0$ and R_1 is the radius of convergence of this power series.

Then:

(a) $\int f(x)dx = \sum_{n=0}^{\infty} \frac{a_n (x-c)^{n+1}}{n+1} + C$ where C is the integration constant and the radius of convergence of this power series is also R_1.

(i.e., $\int (a_0 + a_1(x-c) + a_2(x-c)^2 + ...)dx = \int a_0 dx + \int a_1(x-c)dx + \int a_2(x-c)^2 dx + ...$

$$= a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \frac{a_2}{3}(x-c)^3 + ... + C$$

$$= \sum_{n=0}^{\infty} \frac{a_n (x-c)^{n+1}}{n+1} + C\text{)}$$

(b) If I is the interval of convergence and $a, b \in I$ and $a < b$ and a, b are interior points of I, then,

$$\int_a^b f(x)dx = \sum_{n=0}^{\infty} \left[\frac{a_n (x-c)^{n+1}}{n+1} \right]_a^b = \sum_{n=0}^{\infty} \left[\frac{a_n (b-c)^{n+1}}{n+1} - \frac{a_n (a-c)^{n+1}}{n+1} \right]$$

$$= \sum_{n=0}^{\infty} \frac{a_n (b-c)^{n+1}}{n+1} - \sum_{n=0}^{\infty} \frac{a_n (a-c)^{n+1}}{n+1}$$

(Note:

$\left[\frac{a_n (x-c)^{n+1}}{n+1} \right]_a^b = \int_a^b a_n (x-c)^n dx$ and hence, $\int_a^b (\sum_{n=0}^{\infty} a_n (x-c)^n)dx = \sum_{n=0}^{\infty} (\int_a^b a_n (x-c)^n dx)$).

---

*Note: When the radius of convergence is ∞, $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, $x \in R$ and (a), (b) hold but in (a) the radius of convergence of $\sum_{n=0}^{\infty} \frac{a_n (x-c)^{n+1}}{n+1}$ is ∞ and in (b), $I = R$ and any a, $b \in R$ are always interior points of R.

Examples:

(1) $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...,\ x \in R$

(Note: The radius of convergence of this power series is ∞)

Let $x \in R$.

$$f'(x) = 1 + 2 \times \frac{x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + ...$$
$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$
$$= f(x)$$

* This function is called the <u>exponential function</u> and $f(x)$ is denoted by <u>exp(x)</u>. Also, it is denoted by $\underline{e^x}$.

$$\boxed{\text{So we have } \frac{d}{dx} e^x = e^x \text{ for all } x \in R}$$

(2) $\dfrac{1}{1+x} = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + ...$ where $\underline{-1 < x < 1}$.

(The interval of convergence of $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^n$ is (-1, 1). i.e., the set of all $x \in R$ such that $-1 < x < 1$)

Therefore, $\displaystyle\int \frac{1}{1+x} dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + C$ where $-1 < x < 1$

(<u>Note</u>: Actually this is true for $-1 < x \le 1$)

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + C, \text{ where } C \text{ is the integration constant.}$$

Therefore, $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + K$ for some constant K.

(Since, $\displaystyle\int \frac{1}{1+x} dx = \ln(1+x) + \text{constant}$)

When $x = 0$, we get, $\ln 1 = K$, i.e., $K = 0$.

Therefore, $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - ....$ where $-1 < x < 1$

(<u>Note</u>: Actually this is true for $-1 < x \le 1$)

<u>Theorem 4</u>: Let $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n (x-c)^n$, $c - R_1 < x < c + R_1$ where $R_1 > 0$ and is the radius of convergence of this power series.

Suppose, $\sum_{n=0}^{\infty} a_n (x-c)^n$ is convergent when $x = c + R_1$

(i.e., $\sum_{n=0}^{\infty} a_n R_1^n$ is convergent and $c + R_1 \in I$ where I is the interval of convergence and $c + R_1$ is the right end point of this interval).

Let $b = c + R_1$.

Then, $\lim_{x \to b^-} f(x) = \sum_{n=0}^{\infty} a_n R_1^n$

($\lim_{x \to b^-} f(x)$ means the limit of $f(x)$ as x tends to b but x taking values such that $x < b$)

*A similar result holds for $\lim_{x \to a^+} f(x)$ where $a = c - R_1$.

Example:

We saw that, $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ where $-1 < x < 1$.

By the theorem,

$\lim_{x \to 1^-} \ln(1 + x) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$

i.e., $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$

2.3.2. Taylor and Maclaurin Series

Consider a power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ with radius of convergence R_1 for some $R_1 > 0$ (or radius of convergence ∞).

Then, we have a function $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, $c - R_1 < x < c + R_1$ (or $x \in R$).

We showed that $a_n = \frac{f^n(c)}{n!}$ for all $n \in N$. (See Ex/Result after Theorem 2, in 2.3.1)

i.e., $f(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!} (x-c)^n$, $c - R_1 < x < c + R_1$ (or $x \in R$).

Definition: Now, let us consider functions, not given in terms of power series (example $f(x) = \sin x$, $x \in R$)

We call the series $\sum_{n=0}^{\infty}\frac{f^n(c)}{n!}(x-c)^n$ the Taylor Series for f about c and we call the Taylor Series for f about 0 the Maclaurin Series for f.
(Note: We must have that f is infinitely differentiable at $x = c$)

We give without much details (Note: These details are required in order to apply the theorem) the following theorem:

Theorem: Suppose f is a function defined on an interval I (Note: I could be R) and c is an interior point of I.
Then: $f(x)=\sum_{n=0}^{\infty}\frac{f^n(c)}{n!}(x-c)^n$ for all x such that $c - r_1 < x < c + r_1$ for some $r_1 > 0$ (or for all $x \in R$), i.e., $f(x)$ is equal to its Taylor Series for all x such that $c - r_1 < x < c + r_1$ for some $r_1 > 0$ (or for all $x \in R$).
When $c = 0$, $f(x)=\sum_{n=0}^{\infty}\frac{f^n(c)}{n!}x^n$ (i.e., $f(x)$ is equal to its Maclaurin Series) for all x such that $-r_1 < x < r_1$ for some $r_1 > 0$ (or for all $x \in R$).

Note: Though we have not given the details regarding the function f that are necessary for the theorem, we give the following facts.

(1) Obviously, $f(x)$ can be differentiated infinitely at $x = c$ (i.e., $f^n(c)$ exists for all $n \in N$)

(2) For any $x, y \in (c-r_1, c+r_1)$, $\lim_{n\to\infty}\frac{f^n(y)}{n!}(x-c)^n = 0$ (or for any x, $y \in R$, $\lim_{n\to\infty}\frac{f^n(y)}{n!}(x-c)^n = 0$).
This is a sufficient but not necessary requirement.

Examples:

(1) $f(x) = \sin x,\ x \in R$. Let $x \in R$. Then $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, and $f''''(x) = \sin x$. Therefore for any $n \in N$, $f^{4n-3}(x) = \cos x$, $f^{4n-2}(x) = -\sin x$, $f^{4n-1}(x) = -\cos x$ and $f^{4n}(x) = \sin x$, i.e., $f^{4n-3}(0) = 1$, $f^{4n-2}(0) = 0$, $f^{4n-1}(0) = -1$ and $f^{4n}(0) = 0$. Therefore, for any $x, y \in R$, $\lim_{n\to\infty}\frac{f^n(y)}{n!}x^n = 0$ (since $\left|f^n(y)\right| \le 1$ and hence, $\left|\frac{f^n(y)x^n}{n!}\right| \le \frac{|x|^n}{n!}$ and $\lim_{n\to\infty}\frac{|x|^n}{n!} = 0$).
(Note: $\sum_{n=1}^{\infty}\frac{|x|^n}{n!}$ is convergent and hence $\lim_{n\to\infty}\frac{|x|^n}{n!} = 0$).

(2) $f(x) = (1+x)^r$ where -1 < x < 1.

Here, $r \in R$ but $r \notin N$ and $r \neq 0$.

Let -1 < x < 1. Then, $f'(x) = r(1+x)^{r-1}$, $f''(x) = r(r-1)(1+x)^{r-2}$, etc.,

i.e., $f^n(x) = r(r-1)......(r-n+1)(1+x)^{r-n}$.

Then the Maclaurin Series $\sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n$ is $\sum_{n=0}^{\infty} \frac{r(r-1)....(r-n+1)}{n!}x^n$.

Therefore, the series converges when -1 < x < 1.

For the proof of $(1+x)^r = \sum_{n=0}^{\infty} \frac{r(r-1)...(r-n+1)}{n!}x^n$, see Ref 5: page 34 if you know how to solve differential equations, otherwise ignore it.

This series is called a binomial series.

For further examples/problems, see Ref 5, pages 433 - 439, examples 2, 3, 7 and 8 and problems 1 – 3 (for problem 3 see example 8 on page 423), 4, 5 and 9-17.

To solve some of the problems we state a useful result.

Result: Suppose α is an interior point of the interval of convergence of $\sum_{n=0}^{\infty} a_n x^n$ and β is an interior point of the interval of convergence of $\sum_{n=0}^{\infty} b_n x^n$. Then, $(\sum_{n=0}^{\infty} a_n \alpha^n) \times (\sum_{n=0}^{\infty} b_n \beta^n)$ is equal to the convergent series $\sum_{n=0}^{\infty} c_n$ where for any $n \in N \cup \{0\}$,

$c_n = a_0 b_n \beta^n + (a_1\alpha) \times (b_{n-1}\beta^{n-1}) + (a_2\alpha^2) \times (b_{n-2}\beta^{n-2}) + ... + a_n \alpha^n b_0$

(i.e., $(a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + ...)(b_0 + b_1\beta + +b_2\beta^2 + b_3\beta^3 + ...)$

$= a_0 b_0 + (a_0 \times (b_1\beta)) + (a_1\alpha) \times b_0) + (a_0(b_2\beta^2) + (a_1\alpha)(b_1\beta) + (a_2\alpha^2)b_0) + ...)$

When $\alpha = \beta$, we get, $c_n = \alpha^n (a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + ... + a_n b_0)$ and

$(a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + ...) \times (b_0 + b_1\alpha + +b_2\alpha^2 + b_3\alpha^3 + ...)$

$= a_0 b_0 + \alpha(a_0 b_1 + a_1 b_0) + \alpha^2(a_0 b_2 + a_1 b_1 + a_2 b_0) + \alpha^3(a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0) + \ldots$

*Note: This result could sometimes be used to justify that a function f is equal to its Maclaurin series (we need also the Ex/Result given just before Theorem 3)

-- END --