Face to Face Workshop - Semester 3

IT3305 – Mathematics for Computing II

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CREDITS: 03

Topics	Hours
Matrices	9
Sequences and Series	9
Vectors	9
Differentiation and Integration	9
Basic Statistics	9
Total	45

Matrix

1.1. Definition of a matrix [Ref.1: pg.699, Ref. 4: pg.1]

1.2. Column and row matrices (vectors) [Ref.1: pg.700]

1.3. Square, diagonal, identity, null, symmetric, skew-symmetric, and triangular (upper and lower) matrices, equality of matrices [Ref.1: pg.700-702, Ref.4: pg.2, 10-12, 103 (13.9 and 13.9' only)]

1.4. Matrix addition [Ref.1: pg.702-703, Ref.4: pg.2]

1.5. Scalar multiplication of a matrix [Ref.1: pg.703]

1.6. Matrix multiplication and its properties [Ref.1: pg.712-713, Ref.4: pg. 3]]

1.7. Orthogonal matrix, invertible matrix and transpose of a matrix [Ref.1:pg. 739-742, Ref.4: pg.2, 10-12, 55, 103]

1.8. Determinants of matrices (In particular of orders 2 and 3) and properties of determinants [Ref.1: pg.745-749, Ref.4: pg.20-22]

1.9. Singular and non-singular matrices [Ref.4: pg.39]

1.10. The adjoint of a square matrix and its properties [Ref.1: pg.719-720, Ref.4: pg.11-12, 22-23, 49]

1.11. Finding inverse of a matrix [Ref.1: pg.739-742, Ref.4: pg.55]

1.12. Systems of linear equations (Through examples, where all types and cases are done) [Ref.1: pg.724-739. Ref.4: pg.75-79]

Sequences and Series

2.1 Sequences [Ref.1: pg.266 - 269, Ref.5: pg.385-393]

2.1.1 Definition of a sequence [Ref.1: pg.266, Ref.5: pg.385]

2.1.2 Convergent and divergent sequences [Ref.1: pg.267-268, Ref.5: pg.385]

2.1.3 Limits of a sequence [Ref.1: pg.267-268, Ref.5: pg.385-386]

2.1.4 Elementary properties of limits [Ref.5: pg. 386-387]

2.1.5 Monotonic sequences [Ref.1: pg.268-269, Ref.5: pg.387]

2.1.6 Bounded sequences [Ref.1: pg.268, Ref.5: pg. 386]

2.1.7 Relationship between monotonicity, boundedness [Ref. 5: pg. 388]

2.2 Infinite Series [Ref.1: pg.269-285, Ref.5: pg.394-441]

2.2.1 Definition [Ref.1: pg.269, Ref.5: pg. 394]

2.2.2 Convergence and Divergence [Ref.1: pg.269, Ref.5: pg.394]

2.2.3 Fundamental facts about infinite series [Ref.1: pg. 270-285, Ref.5: pg.395-418]

2.3 Power Series [Ref.5: pg.419-441]

2.3.1 Fundamental facts about power series [Ref.5: pg. 419-431]

2.3.2 Taylor and Maclaurin Series [Ref.5: pg.432-441]

Vectors

3.1 Definition of a vector and a scalar [Ref.1: pg.762, Ref. 3: pg.2-3]

3.2 Equality of vectors [Ref.1: pg. 763, Ref.3: pg.3]

3.3 Geometric representation of a vector [Ref.1: pg.762, Ref.3: pg.2]

3.4 Magnitude of a vector [Ref.1: pg.763, Ref.3: pg.4]

3.5 Unit vector and null vector [Ref.1: pg.763, Ref.3: pg.5]

3.6 Multiplication of a vector by a scalar [Ref.1: pg.766, Ref.3: pg.5-7]

3.7 Vector addition and subtraction [Ref.1: pg.764-766, Ref.3: pg.7-11]

3.8 Position vectors [Ref.1: pg.770-771, Ref. 3: pg.25-26]

3.9 Vectors in a plane and in 3-dimensional space [Ref.1:pg. 771-772]

3.10 The angle between two vectors [Ref.1: pg.164, 772-773]

3.11 The Ratio Theorem [Ref.1: pg.774-775, Ref.3: pg.26-27]

3.12 Scalar product, vector product, and their properties [Ref.1: pg.779-795, Ref.3: pg.70-75, 116-120]

Differentiation and Integration

4.1 Differentiation [Ref.1: pg.545-548, 551-561, 563-565, 567-568, 577-579, 596-597, 606-612, Ref.5: pg.61-63, 71-73, 79-80, 86-87, 89, 102, 108, 110, 115-117, 129-130, 153, 155-156, 166-169, 175-176, 225, 234-235, 237, 243-245],

4.1.1 Definition

4.1.2 Properties and examples

4.1.3 Higher order derivatives

4.1.4 Finding limits using L'Hospital's Rule

4.2 Integration [Ref.1: pg.630-641, 649-651, 657-661, 672-675, Ref.2: pg.196-198, 206-211, 216—218, 225, 234-238, 257-259, 281-309]

4.2.1 Integration as the inverse of differentiation (the indefinite integral)

4.2.2 Integration of standard functions (e^x, log x, sin x, cos x, tan x, sec x, cosec x, cot x)

4.2.3 Properties of integration

4.2.4 Techniques of integration

4.2.5 Area under a curve (the definite integral)

Basic Statistics

5.1 Random variables [Ref.2: pg.36-40, 47-49, 68-69]

5.1.1 Discrete random variables

5.1.2 Continuous random variables

5.2 Probability distribution of a discrete random variable (Ref.2: pg.47)

5.2.1 Definition

5.2.2 Mean and Variance

5.3 The Binomial probability distribution [Ref.2: pg.113-115]

5.4 The Poisson probability distribution [Ref.2: pg.116-117]

5.5 Probability distribution of a continuous random variable [Ref.2: pg.49,113-119]

5.5.1 Definition

5.5.2 Mean and Variance

5.5.3 The Uniform probability distribution

5.5.4 The Normal probability distribution

5.5.5 Normal approximation of the Binomial distribution

5.5.6 The Exponential distribution

Matrix

A matrix is an array of $m \times n$ elements arranged in m rows and n columns. Such a matrix A is usually denoted by.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ .... & \dots & \dots & .. \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m\times n} = [a_{ij}]_{m\times n}$$

a_{11}, a_{12},….., a_{mn} are called the **elements** of the matrix

Determinant of Matrix

Every square matrix A is associated with a scalar called the **determinant of A**, and is denoted by |A|.

Let $A = (a_{ij})$ be a square matrix of order one. Then we define $\mathbf{|A| = a_{11}}$

Let $A = (a_{ij})$ be a square matrix of order two. Then we define $\mathbf{|A| = a_{11}a_{22} - a_{12}a_{21}}$.

Minors of the Matrix

- The **minor** M_{ij} of the element a_{ij} of A_{nn} is the determinant of order n -1 matrix obtained by deleting the row and column containing a_{ij}.

If $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$ Then $M_{23} = \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} = 2$

$M_{33} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -1$

Cofactors of the Matrix

Let $A = (a_{ij})$ be a square matrix of order n. The **cofactor** C_{ij} of a_{ij} is defined as $C_{ij} = (-1)^{i+j}M_{ij}$

$$\text{If} \quad A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$$

$$C_{23} = (-1)^{2+3}M_{23} = (-1)(0) = 0$$

$$C_{33} = (-1)^{3+3}M_{33} = (+1)(1) = 1$$

Determinant of the Matrix A with Order 3

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}.$$

$$\text{If } A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$$

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}.$$

$$= 1x(+1)M_{11} + 0x(-1)M_{12} + 2x(+1)M_{13}$$

$$= 1x(+1)(-1) + 0x(-1)(0) + 2x(+1)(1)$$

$$= -1 + 0 + 2 = 1$$

Ex. Calculate the determinant of $A = \begin{pmatrix} 2 & 3 & 4 \\ -5 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

expanding along

(a) The first row

(b) The first column

(c) The second column

Ex. If $A = \begin{bmatrix} 1 & -4 & 2 & -2 \\ 4 & 7 & -3 & 5 \\ 3 & 0 & 8 & 0 \\ -5 & -1 & 6 & 9 \end{bmatrix}$, Find |A|.

Nonsingular and Singular Matrix

- If $|A| \neq 0$, then A is said to be a **nonsingular** (or **invertible**) matrix; otherwise it is said to be **singular**.

Note: If A^{-1} exists, $|A^{-1}| = 1/ |A|$

Let A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

If $|A| \neq 0$ then

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example :

Let A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Then |A| = $1 \times 4 - 2 \times 3 = -2 \neq 0$. Therefore, A^{-1} exists and

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

- **Properties of determinants**
  - Let A be a matrix of order n. Then $|A^T| = |A|$.

Example :

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$$

  - Let A be a matrix of order n. Then $|kA| = k^n|A|$ where k is a scalar .

Example :

$$\left| 2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right| = \begin{vmatrix} 2 & 4 \\ 6 & 8 \end{vmatrix} = 2^2 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

– Let A be a matrix of order n. If any two rows (or columns) of A are identical, then $|A| = 0$.

Example :

$$\begin{vmatrix} 1 & 3 & 1 \\ 2 & 4 & 2 \\ 3 & 5 & 3 \end{vmatrix} = 0$$

– If $A = (a_{ij})$ is a diagonal matrix or a triangular matrix of order n, then $|A| = a_{11}\, a_{22} \ldots\ldots a_{nn}$.

Example :

$$\begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 6 \end{vmatrix} = 1.2.6 = 12$$

– Let I be the identity matrix of order n. Then $|I| = 1$.

Example :

$$\text{Let } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|I| = 1(1-0) - 0(0 - 0) + 0(0-0) = 1$$

– Let A be a matrix of order n. If B is obtained from A by interchanging any two rows (or columns) of A, then |B| = - |A|.

Example :

$$\begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 6 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 3 \\ 0 & 0 & 6 \\ 0 & 2 & 8 \end{vmatrix}$$

– Let A be a matrix of order n. If B is obtained from A by multiplying a row (or column) by a nonzero scalar k, then |B| = k|A|.

$$\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix}$$

– Let A be a matrix of order n. If B is obtained from A by adding a scalar multiple of a row (or column) of A to another row (or column) of A, then |B| = |A|.

Example :

$$\begin{vmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 6 \end{vmatrix} = \begin{vmatrix} 1+0.k & 1+2k & 4+5k \\ 0 & 2 & 5 \\ 0 & 0 & 6 \end{vmatrix}$$

– If any row (or column) of A is the sum of two or more elements, then the determinant can be expressed as the sum of two or more determinants.

Example :

$$\begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1+3 \\ 2 & 3 & 2+3 \\ 3 & 5 & 4+2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 5 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \\ 3 & 5 & 2 \end{vmatrix}$$

Let A be a matrix of order n. If B is also a square matrix of order n, then $|AB| = |A||B|$.

We note from the above result that if A is invertible then since $A.A^{-1} = I$,

- $|A.A^{-1}| = |A||A^{-1}| = |I| = 1.$
- Thus, $|A^{-1}| = 1/ |A|$.
- We can conclude that if $|A| = 0$, the inverse of A does not exist.

Adjoint of a Matrix

adj A = C^T

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 0 \\ 1 & -1 & 1 \end{pmatrix} \quad C^T = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

If A is a non-singular, $A^{-1} = \frac{1}{|A|} adj\, A$

$|A| = 3$ $\qquad$ $A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -1 & 0 & 1 \end{pmatrix}$

Systems of Linear Equations

$$3\,x_1 + x_2 = 9$$
$$5\,x_1 - 3\,x_2 = 1$$

$$\begin{pmatrix} 3 & 1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$$

$$AX = B$$

Systems of Linear Equations

2x – 3y + 6z = -18

6x + 4y – 2z = 44

5x + 8y + 10z = 56

$$\begin{pmatrix} 2 & -3 & 6 \\ 6 & 4 & -2 \\ 5 & 8 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -18 \\ 44 \\ 56 \end{pmatrix}$$

AX = B

Systems of Linear Equations

A system of **m** linear equations in **n** unknowns is of the form

$$A_{11}\, x_1 + A_{12}\, x_2 + \ldots\ldots + A_{1n}\, x_n = y_1$$
$$A_{21}\, x_1 + A_{22}\, x_2 + \ldots\ldots + A_{2n}\, x_n = y_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_{m1}\, x_1 + A_{m2}\, x_2 + \ldots\ldots + A_{mn}\, x_n = y_m$$

where $y_1, y_2 \ldots\ldots y_m$ and A_{ij} $1 \le i \le m$, $1 \le j \le n$ are real numbers and $x_1, x_2, \ldots x_n$ are n unknowns.

Note: If $y_1 = y_2 = \ldots\ldots = y_m = 0$, the system is called a **homogeneous** system.

We write this system in matrix form as:

$$\begin{pmatrix} A_{11} & A_{12} & . & . & A_{1n} \\ A_{21} & A_{22} & . & . & A_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ A_{m1} & A_{m2} & . & . & A_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_m \end{pmatrix}$$

$$AX = Y$$

A is called the **matrix of coefficients** of the system.

Note: Y is zero (zero matrix), the system is called a **homogeneous** system.

- A solution of the system of linear equations is a set of values $x_1, x_2, \ldots x_n$ which satisfy the above m equations.

- If the equations are homogeneous then, $x_1 = x_2 = \ldots. = x_n = 0$ is a solution of the system.

- If the system is not homogeneous, it is possible that no set of values will satisfy all the equations in the system.

- If this is the case the system is said to be **inconsistent**.

- If there exists a solution which satisfies all the equations of the system, the system is said to be **consistent**.

- A homogeneous system is always consistent since it has the trivial **solution** $x_1 = x_2 = \ldots\ldots = 0$.

- There are two possible types of solutions to a consistent system of linear equations.
  - Either the system will have a unique solution, or
  - it will have infinitely many solutions.

- If a homogeneous system has a unique solution then, since the trivial solution is always a solution, the trivial solution will be its unique solution.

An example of a system which has a unique solution is

$$2x + y = 5$$

$$x - y = 4.$$

The solution to this system is $x = 3$, $y = -1$.

An example of a system which has infinitely many solutions is:

$$2x + 3y + 4z = 5$$
$$x + 6y + 7z = 3$$

This system has infinitely many solutions of the form $x = k$, $y = (10k - 23) / 3$ and $z = (7 – 3k)$ where k is any scalar.

Elementary Row Operations

1. Any two rows of a matrix may be interchanged.

2. A row may be multiplied by a nonzero constant

3. A multiple of one row may be added to another row

Example:

x + 2y –3z = -1

3x – y + 2z = 7

5x + 3y – 4z = 2

This system in matrix form

$$\begin{pmatrix} 1 & 2 & -3 \\ 3 & -1 & 2 \\ 5 & 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$$

Multiplying the first row by –3 and adding it to the second row we obtain .

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -7 & 11 \\ 5 & 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \\ 2 \end{pmatrix}$$

- Multiplying the first row by –5 and adding it to the third row we obtain.

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -7 & 11 \\ 0 & -7 & 11 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \\ 7 \end{pmatrix}$$

- Multiplying row 2 by –1 and adding to row 3 we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -7 & 11 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \\ -3 \end{pmatrix}$$

Thus the system reduces to

$x + 2y - 3z = -1$

$-7y + 11z = 10$

$0 = -3$

This shows that the system is **inconsistent** since the third equation is false. Thus this system has no solution.

Example :

x + 2y – 3z = 6

2x - y + 4z = 2

4x + 3y - 2z = 14

This set of equations in matrix form

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 14 \end{pmatrix}$$

- Multiplying row 2 by –2 and adding to row 3 we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 0 & 5 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 10 \end{pmatrix}$$

- Multiplying row 1 by –2 and adding to row 2 we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 10 \\ 0 & 5 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \\ 10 \end{pmatrix}$$

- Adding row 2 to row 3 we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 10 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \\ 0 \end{pmatrix}$$

- Multiplying row 2 by -1/5 we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix}$$

- Multiplying row 2 by –2 and adding to row 1 we obtain

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

Thus the system reduces to

x + z = 2
y – 2z = 2
0 = 0

This system is consistent and has infinitely many solutions given by x = k, y = 6 – 2k, z = 2 - k, where k is a scalar.

Example:

2x + y + 3z = 5

3x – 2y +2z = 5

5x – 3y – z = 16

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & -2 & 2 \\ 5 & -3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 16 \end{pmatrix}$$

Adding row 1 to row 2 we obtain

$$\begin{pmatrix} 2 & 1 & 3 \\ 5 & -1 & 5 \\ 5 & -3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 16 \end{pmatrix}$$

- Multiplying row 2 by –1 and adding to row 3 we obtain

$$\begin{pmatrix} 2 & 1 & 3 \\ 5 & -1 & 5 \\ 0 & -2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 6 \end{pmatrix}$$

- Multiplying row 2 by 1/5 and then multiplying row 3 by –1/2 we obtain

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & -1/5 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}$$

- Multiplying row 3 by –1 and adding to row1 we obtain

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & -1/5 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ -3 \end{pmatrix}$$

- Multiplying row 1 by ½ we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1/5 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$$

- Multiplying row 1 by –1 and adding to row 2 and then multiplying row 3 by 1/5 and adding to row 2 we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 8/5 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -13/5 \\ -3 \end{pmatrix}$$

- Multiplying row 2 by 5/8 we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -13/8 \\ -3 \end{pmatrix}$$

- Multiplying row 2 by -3 and adding to row 3 we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -13/8 \\ 15/8 \end{pmatrix}$$

- Finally interchanging row 2 and row 3 we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 15/8 \\ -13/8 \end{pmatrix}$$

Thus the system is reduced to

x = 4, y = 15/8 and z = -13/8 and this is the unique solution to the system.

- **Result :** Suppose a system of linear equations in matrix form is AX = Y. If the matrix A is invertible, the system has a unique solution given by $X = A^{-1}Y$.

Example :

Consider the following system of linear equations.

$x - z = 3$

$y + z = 3$

$x + 2z = 6.$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}$$

The matrix A = $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ has $|A| \neq 0$

Thus A^{-1} must exist

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

- Thus the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

- Therefore we obtain the unique solution x = 4, y = 2 and z =1

2x + y – 2z = -1
4x - 2y + 3z = 14
x – y + 2z = 7

$$\begin{pmatrix} 2 & 1 & -2 \\ 4 & -2 & 3 \\ 1 & -1 & 2 \end{pmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 14 \\ 7 \end{bmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & -2 & -1 \\ 4 & -2 & 3 & 14 \\ 1 & -1 & 2 & 7 \end{array}\right)$$

2x + 1y – 2z = -1
4x - 2y + 3z = 14
1x – 1y + 2z = 7

r(1) = r(1) + r(3)

$$\left(\begin{array}{ccc|c} 3 & 0 & 0 & 6 \\ 4 & -2 & 3 & 14 \\ 1 & -1 & 2 & 7 \end{array}\right)$$

3x + 0y + 0z = 6
4x - 2y + 3z = 14
1x – 1y + 2z = 7

r(1) = (1/3) x r(1)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 4 & -2 & 3 & 14 \\ 1 & -1 & 2 & 7 \end{array}\right)$$

1x + 0y + 0z = 2
4x - 2y + 3z = 14
1x – 1y + 2z = 7

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 4 & -2 & 3 & 14 \\ 1 & -1 & 2 & 7 \end{array}\right)$$

1x + 0y + 0z = 2
4x - 2y + 3z = 14
1x – 1y + 2z = 7

r(2) = r(2) - 4 x r(1)
r(3) = r(3) - r(1)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -2 & 3 & 6 \\ 0 & -1 & 2 & 5 \end{array}\right)$$

1x + 0y + 0z = 2
0x - 2y + 3z = 6
0x – 1y + 2z = 5

r(2) = -(1/2) x r(2)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -3/2 & -3 \\ 0 & -1 & 2 & 5 \end{array}\right)$$

1x + 0y + 0z = 2
0x + 1y – (3/2)z = -3
0x – 1y + 2z = 5

r(3) = r(2) + r(3)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -3/2 & -3 \\ 0 & 0 & 1/2 & 2 \end{array}\right)$$

1x + 0y + 0z = 2
0x + 1y – (3/2)z = -3
0x + 0y +(1/2)z = 2

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -3/2 & -3 \\ 0 & 0 & 1/2 & 2 \end{array}\right)$$

$$\begin{aligned} 1x + 0y + \quad 0z &= 2 \\ 0x + 1y - (3/2)z &= -3 \\ 0x + 0y + (1/2)z &= 2 \end{aligned}$$

r(3) = 2 x r(3)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -3/2 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

$$\begin{aligned} 1x + 0y + \quad 0z &= 2 \\ 0x + 1y - (3/2)z &= -3 \\ 0x + 0y + \quad 1z &= 4 \end{aligned}$$

r(2) = r(2) + (3/2) x r(3)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

$$\begin{aligned} 1x + 0y + 0z &= 2 \\ 0x + 1y + 0z &= 3 \\ 0x + 0y + 1z &= 4 \end{aligned}$$

x = 2, y = 3, z = 4

1) Which of the following is/are true about a diagonal matrix?

(a) It is always a square matrix.
(b) No element along the diagonal is equal to zero.
(c) It is always an upper triangular matrix.
(d) It is always an identity matrix.
(e) It is always a symmetric matrix.

2) Find A^2 - $2B$ where $A = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & -6 \\ 12 & 8 \end{bmatrix}$.

(a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

3) Let $A = \begin{bmatrix} 3 & 2 & 2 & -2 \\ 12 & 2 & 12 & 2 \\ 11 & 0 & 11 & 0 \\ 21 & 0 & 21 & 1 \end{bmatrix}$. Then |A| is equal to

(a) 11 (b) -11 (c) 22
(d) 0 (e) -22

4) If A=$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and B^{-1}=$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -1 & 0 & 2 \end{bmatrix}$, then find $((BA)^T)^{-1}$

(a) $\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 0 & 1 \\ 1 & 6 & -1 \\ -1 & 0 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & 1 & -1 \\ 0 & 6 & 0 \\ 1 & -1 & 4 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 1/3 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & -1/3 & 1 \end{bmatrix}$

5) Let A=$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$. Find $(\text{adj } A)^T$

(a) $\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$ (b) $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ (c) $\begin{bmatrix}4 & -3\\ -2 & 1\end{bmatrix}$ (d) $\begin{bmatrix}-4 & 3\\ 2 & -1\end{bmatrix}$ (e) $\begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix}$

6) Let $A = (a_{ij})$ be an upper triangular matrix of order n. Which of the following must be true about A?

(a) $a_{ij} = 0$ whenever $i < j$, where $i, j \in \{1, 2, \ldots, n\}$.
(b) $a_{ij} = 0$ whenever $i > j$, where $i, j \in \{1, 2, \ldots, n\}$.
(c) All the entries above the diagonal are zero.
(d) $a_{ij} \neq 0$ whenever $i = j$, where $i, j \in \{1, 2, \ldots, n\}$.
(e) All the entries below the diagonal are zero.

Basic Statistics

Learning Objectives

- Define what random variables are and how they are used.
- Define what is meant by a discrete probability distribution.
- Compute the mean and variance of a discrete and a continuous random variable
- Use and interpret some discrete probability distributions such as Binomial and Poisson
- Use and interpret some continuous probability distributions such as the Uniform, Normal & Exponential
- Identify properties of the Normal probability distribution
- Compute normal probabilities using standard normal tables
- Convert a random variable to a standard normal random variable
- Use the normal probability distribution to approximate binomial probabilities

DETAILED SYLLABUS

- **Random variables**
  - Discrete random variables
  - Continuous random variables
- **Probability distribution of a discrete random variable**
  - Definition
  - Mean and Variance
  - The Binomial probability distribution
  - The Poisson probability distribution
- **Probability distribution of a continuous random variable**
  - Definition
  - Mean and Variance
  - The Uniform probability distribution
  - The Normal probability distribution
  - Normal approximation of the Binomial distribution
  - The Exponential distribution

Some Questions on Past Papers

Which one of these variables is a discrete random variable?

- Your national identity card number without English letter
- Your island rank at the G.C.E. (A/L) examination
- Number of questions completed by you at the end of the allocated time period in an examination.
- Number of women taller than 68 inches in a random sample of 50 men.
- Downloaded size in Kilo-bites (Kb) of a MP3 file.

Consider the following three random variables.

- **X: The number of tattoos a randomly selected person has.**
- **Y: The outside temperature today.**
- **Z :The number of women taller than 68 inches in a random sample of 10 women.**

Which is the correct about the type of variables?

- X : Discrete, Y : Continuous, Z : Continuous
- X : Discrete, Y : Discrete, Z : Continuous
- X : Discrete, Y : Continuous, Z : Discrete
- X : Discrete, Y : Discrete, Z : Discrete
- X : Continuous, Y : Continuous, Z : Discrete

The mean and variance of a binomial distribution are 10 and 8 respectively. What are the parameters of this distribution?

– $n = 10$, $p = 0.8$
– $n = 10$, $p = 0.2$
– $n = 50$, $p = 0.2$
– $n = 50$, $p = 0.8$
– $n = 100$, $p = 0.2$

If the standard deviation of a Poisson distribution is 2 then the mean of it;

- 0.25
- 0.5
- 1.41
- 2
- 4

- The number of virus alerts in a day of a particular computer is a discrete random variable with the following probability distribution function.

X	1	2	3	4	5
Probability	2a	0.25	0.4	a	0.05

a). Determine the value of a.

b). Calculate the probability $P(1 < X \leq 3)$.

c). Calculate the probability $P(X > 2)$.

d). Calculate $E(X)$.

e). Calculate $V(X)$.

- Answer

The time taken to download a certain type of virus guard follows a normal distribution with mean *72 seconds* and variance *36 seconds*.

- Calculate the probability that the time taken to download that type of virus guard is more than *75 seconds*.
- Calculate the probability that the time taken to download that type of virus guard is between *72 seconds* and *75 seconds*.
- Only 95% of time taken to download that type of virus guard is less than how many seconds?

- Answer

Let X be the continuous random variable which is defined as '***the length of a random access memory (RAM) card***', in centimeters (cm), produced by a particular manufacturer. The probability density function (pdf) of X is given as follows.

$$f(x) = \begin{cases} kx & 0 \leq x \leq 6 \\ 0 & \textit{otherwise} \end{cases}$$

(i) Calculate the value of k.
(ii) Calculate the probability that the length of a RAM card is not more than 5 cm.
(iii) Calculate the probability that the length of a RAM card is at least 5 cm.
(iv) Evaluate the expected value of X.
(v) Evaluate the standard deviation of X.

Answer