Integration

References: (1) Business Mathematics by Qazi Zameeruddin, V. K. Khanna and S.K. Bhambri
(2) Schaum's Outlines, Calculus by Frank Ayres, Jr. and Elliot Mendelson (4th Edition)

Integration or antidifferentiation is the reverse process of differentiation. It is the process of finding an original function when the derivative of the function is given.

1. Antiderivative and the Indefinite Integral (Ref 1: pg. 630-632, Ref 2: pg. 196-198)

Definition:
A function $F(x)$ is called an **antiderivative** of the function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of $f(x)$.

Example:
(1) If $f(x) = 3x^2$ then $F(x) = x^3$ is an antiderivative of $f(x)$ since $F'(x) = 3x^2 = f(x)$.
(2) If $f(x) = \cos x$ then $F(x) = \sin x$ is an antiderivative of $f(x)$ since $F'(x) = \cos x = f(x)$.
(3) If $f(x) = x^4$ then $F(x) = \dfrac{x^5}{5}$ is an antiderivative of $f(x)$ since $F'(x) = x^4 = f(x)$.

Example:
Consider the following functions and their derivatives.

(i) If $F_1(x) = x^3$, then $F_1'(x) = 3x^2$
(ii) If $F_2(x) = x^3 + 5$, then $F_2'(x) = 3x^2$
(iii) If $F_3(x) = x^3 - 10$, then $F_3'(x) = 3x^2$
(iv) If $F_4(x) = x^3 + \sqrt{2}$, then $F_4'(x) = 3x^2$

Therefore, if $f(x) = 3x^2$, then $F_1(x) = x^3$, $F_2(x) = x^3 + 5$, $F_3(x) = x^3 - 10$ and $F_4(x) = x^3 + \sqrt{2}$ are all antiderivatives of $f(x) = 3x^2$.

We see from the above example, that the antiderivative of a function is not unique. If $F(x)$ is an antiderivative of $f(x)$, then $F(x) + c$ (where c is a constant) is also an antiderivative of $f(x)$, since $\dfrac{d}{dx}[F(x) + c] = F'(x)$.

Theorem:
If $F_1(x)$ and $F_2(x)$ are antiderivatives of a function f, then $F_1(x) - F_2(x) = C$, where C is a constant.

It follows from the above theorem that if $F_1(x)$ is an antiderivative of a function $f(x)$, then every other antiderivative of $f(x)$ is given by $F(x) = F_1(x) + C$, where C is an arbitrary constant. C is called the **constant of integration**.

We denote the process of integration by the symbol $\int$ which is called an **integral sign**. The expression $\int f(x)dx$ denotes any antiderivative of $f(x)$; i.e., if $F'(x) = f(x)$ then $\int f(x)dx = F(x) + C$. $\int f(x)dx$ is also called the **indefinite integral of *f*(*x*)**. In this notation, $f(x)$ is called the **integrand** of the indefinite integral and the x in dx is the **variable of integration**. $\int f(x)dx$ is read as 'the integral of $f(x)$ with respect to x'.

Since differentiation is the inverse operation of integration,

$$\frac{d}{dx}[\int f(x)dx] = f(x)$$

And since integration is the inverse operation of differentiation,

$$\int f'(x)dx = f(x) + C\,.$$

<u>Rules of Integration</u>

(1) $\int kdx = kx + C$ (k a constant) [since $\frac{d}{dx}(kx + C) = k$]

(2) $\int kf(x)dx = k\int f(x)dx$ (k a constant)

(3) $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

(4) $\int x^r dx = \frac{x^{r+1}}{r+1} + C\,, \quad (r \neq -1)$ [since $\frac{d}{dx}(\frac{x^{r+1}}{r+1} + C) = x^r$]

(5) $\int (g(x))^r g'(x)dx = \frac{(g(x))^{r+1}}{r+1} + C\,, \quad (r \neq -1)$

[since $\frac{d}{dx}[\frac{(g(x))^{r+1}}{r+1} + C] = (g(x))^r g'(x)$]

<u>Example</u>:

(1) $\int \sqrt{3}dx = \sqrt{3}x + C$ by applying (1) above

(2) $\int 4x^5 dx = 4\frac{x^6}{6} + C = \frac{2x^6}{3} + C$ by applying (2) and (4) above

(3) $\int [2x^{\frac{1}{2}} + 10]dx = 2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 10x + C = \frac{4x^{\frac{3}{2}}}{3} + 10x + C$ by applying (1), (2), (3) and

(4) above

(4) $\int 4(4x+1)^2 dx = \dfrac{(4x+1)^3}{3} + C$ by applying (5) above

2. **Table of Indefinite Integrals** (Ref 1: pg. 630-631, Ref 2: pg. 198, 225-228, 234 – 237)

1. $\int x^r dx = \dfrac{x^{r+1}}{r+1} + C \quad (r \neq -1)$ since	$\dfrac{d}{dx}\left(\dfrac{x^{r+1}}{r+1} + C\right) = x^r$
2. $\int \dfrac{1}{x} dx = \ln\lvert x\rvert + C \quad (x \neq 0)$ since $\int \dfrac{g'(x)}{g(x)} dx = \ln\lvert g(x)\rvert + C$	$\dfrac{d}{dx}(\ln\lvert x\rvert + C) = \dfrac{1}{x}$
3. $\int a^x dx = \dfrac{a^x}{\ln a} + C \quad (0 < a,\ a \neq 1)$ since	$\dfrac{d}{dx}\left(\dfrac{a^x}{\ln a} + C\right) = a^x$
4. $\int e^x dx = e^x + C$ since	$\dfrac{d}{dx}(e^x + C) = e^x$
5. $\int \sin x dx = -\cos x + C$ since	$\dfrac{d}{dx}(-\cos x + C) = \sin x$
6. $\int \cos x dx = \sin x + C$ since	$\dfrac{d}{dx}(\sin x + C) = \cos x$
7. $\int \sec^2 x dx = \tan x + C$ since $(x \neq \dfrac{\pi}{2} + k\pi,\ k = 0, \pm 1, \pm 2, ...)$	$\dfrac{d}{dx}(\tan x + C) = \sec^2 x$
8. $\int \operatorname{cosec}^2 x dx = -\cot x + C$ since $(x \neq k\pi,\ k = 0, \pm 1, \pm 2, ...)$	$\dfrac{d}{dx}(-\cot x + C) = \operatorname{cosec}^2 x$
9. $\int \tan x dx = -\ln\lvert\cos x\rvert + C \quad (x \neq \dfrac{\pi}{2} + k\pi,\ k = 0, \pm 1, \pm 2, ...)$	
10. $\int \cot x dx = \ln\lvert\sin x\rvert + C \quad (x \neq k\pi,\ k = 0, \pm 1, \pm 2, ...)$	
11. $\int \operatorname{cosec} x dx = \ln\lvert\operatorname{cosec} x - \cot x\rvert + C$	
12. $\int \sec x dx = \ln\lvert\sec x + \tan x\rvert + C$	
13. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\dfrac{x}{a} + C = -\cos^{-1}\dfrac{x}{a} + C \quad (\lvert x\rvert < a)$	
14. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C = -\dfrac{1}{a}\cot^{-1}\dfrac{x}{a} + C \quad (a > 0)$	
15. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\dfrac{x}{a} + C \quad (x > a > 0)$	

3. **The Substitution Method** (Ref 1: pg. 637-638, 640, Ref 2: pg. 198, 289-294)

$$\int f(g(x))g'(x)dx = \int f(u)du$$

where u is replaced by $g(x)$ after the right-hand side is evaluated.

Justification:

Let $u = g(x)$. Then $\frac{du}{dx} = g'(x)$.

$$\frac{d}{dx}\left(\int f(u)du\right) = \frac{d}{du}\left(\int f(u)du\right)\frac{du}{dx} \quad \text{by applying the chain rule}$$

$$= f(u).\frac{du}{dx} \quad \text{since differentiation is the reverse of integration}$$

$$= f(g(x))g'(x)$$

Therefore, $\int f(u)du = \int f(g(x))g'(x)dx$ by the definition of the antiderivative.

Example:

(1) $\int \tan x dx = \int \frac{\sin x}{\cos x}dx$.

Let $u = \cos x$. Then $\frac{du}{dx} = -\sin x$

Therefore, $\int \frac{\sin x}{\cos x}dx = \int \frac{-1}{u}du$ by substitution

$\int -\frac{1}{u}du = -\ln|u| + C$

Therefore, $\int \tan x dx = -\ln|\cos x| + C$ by substituting back $u = \cos x$.

(2) $\int \frac{x^2}{2x^3+5}dx$

Let $u = 2x^3 + 5$. Then $\frac{du}{dx} = 6x^2$. Therefore, $\frac{1}{6}\frac{du}{dx} = x^2$

Thus $\int \frac{x^2}{2x^3+5}dx = \frac{1}{6}\int \frac{1}{u}du = \frac{1}{6}\ln|u| + C = \frac{1}{6}\ln|2x^3+5| + C$

(3) $\int (\cos x + 4)^3 \sin x dx$

Let $u = \cos x + 4$. Then $\frac{du}{dx} = -\sin x$.

Therefore, $\int (\cos x + 4)^3 \sin x dx = \int -u^3 du = -\frac{u^4}{4} + C = -\frac{(\cos x + 4)^4}{4} + C$

(4) $\int \frac{1}{x-a} dx$ where a is a constant.

Let $u = x - a$. Then $\frac{du}{dx} = 1$.

$\int \frac{1}{x-a} dx = \int \frac{1}{u} du = \ln|u| + C = \ln|x-a| + C$.

4. **Integration by Parts** (Ref 1: pg. 649-651, Ref 2: pg. 281-283)

$$\boxed{\int (uv')dx = uv - \int (vu')dx}$$

Justification:

$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x)$

Therefore, $\int (u(x)v(x))'dx = \int [u'(x)v(x) + u(x)v'(x)]dx$

$= \int u'(x)v(x)dx + \int u(x)v'(x)\,dx$

i.e., $u(x)v(x) = \int v(x)u'(x)dx + \int u(x)v'(x)dx$

Therefore, $\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$

Another way of writing this is:

$$\boxed{\int udv = uv - \int vdu}$$

Justification:

$\frac{d}{dx}[\int udv] = \frac{d}{dv}[\int udv].\frac{dv}{dx}$ by the chain rule.

Also, $\frac{d}{dv}\int udv = u$ and hence $\frac{d}{dv}[\int udv]\frac{dv}{dx} = u.v'$.

Thus $\frac{d}{dx}[\int udv] = \frac{d}{dv}[\int udv]\frac{dv}{dx} = uv'$.

Hence $\int udv = \int uv'dx$.

Similarly $\int vdu = \int vu'dx$

Therefore, $\int uv'dx = uv - \int vu'dx$ may also be written as $\int udv = uv - \int vdu$

This technique enables us to replace the 'harder' integration $\int udv$ by an 'easier' integration $\int vdu$.

<u>Example</u>:

(1) $\int xe^{2x}dx$

Let $u = x$ and $dv = e^{2x}dx$.

Then $du = dx$ and $v = \int dv = \int e^{2x}dx = \frac{e^{2x}}{2}$

Therefore, by applying $\int udv = uv - \int vdu$ we obtain

$$\int xe^{2x}dx = x\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

(2) $\int \ln x dx$

Let $u = \ln x$ and let $dv = dx$.

Then $\frac{du}{dx} = \frac{1}{x}$ and $v = x$

Therefore, $\int \ln x dx = x\ln x - \int x.\frac{1}{x}dx = x\ln x - \int dx = x\ln x - x + C$

(3) $\int \cos^3 x dx = \int \cos^2 x \cos x dx$

Let $u = \cos^2 x$ and $dv = \cos x dx$

Then $\frac{du}{dx} = -2\cos x \sin x$ and $v = \int dv = \int \cos x dx = \sin x$

Therefore, $\int \cos^3 x dx = \int \cos^2 x \cos x dx$

$$= \cos^2 x \sin x - \int \sin x(-2\cos x \sin x)dx$$

$$= \cos^2 x \sin x + 2\int \cos x(1 - \cos^2 x)dx$$

$$= \cos^2 x \sin x + 2\int \cos x dx - 2\int \cos^3 x dx$$

Therefore, $3\int \cos^3 x dx = \cos^2 x \sin x + 2\int \cos x dx$

Thus $\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$.

Note: The integral $\int \cos^3 x dx = \int \cos^2 x \cos x dx = \int (1 - \sin^2 x)\cos x dx$ may also be found by making the substitution $u = \sin x$

5. **Partial Fractions** (Ref 2: pg. 304-309)

A function of the form $\frac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials is called a **rational function**. $N(x)$ is the **numerator** and $D(x)$ is the **denominator**. Suppose the degree of $N(x)$ is n and the degree of $D(x)$ is m. If $n < m$, then $\frac{N(x)}{D(x)}$ is said to be a proper fraction. If $n \geq m$, $\frac{N(x)}{D(x)}$ is called an improper fraction.

An improper fraction can be expressed as the sum of a polynomial and a proper fraction; for, if $n \geq m$, we can divide the numerator by the denominator, obtaining as quotient a polynomial $Q(x)$ of degree $n - m$, and as remainder a polynomial $R(x)$ of degree not greater than $m - 1$.

$$\frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}.$$

A polynomial is said to be **irreducible**, if it cannot be written as a product of two polynomials of lower degree. Any linear polynomial $f(x) = ax + b$ is automatically irreducible.

If we consider a quadratic $g(x) = ax^2 + bx + c$, this is irreducible if and only if $b^2 - 4ac < 0$.

<u>Theorem</u>:
Any polynomial $D(x)$ can be written as a product of linear factors and irreducible quadratic factors.

<u>Partial Fractions</u>

A rational function of the form $\frac{cx+d}{(x-a)(x-b)}$ where $a \neq b$ may be expressed as the sum of two fractions of the form $\frac{A}{(x-a)}$ and $\frac{B}{(x-b)}$. These two fractions are called the **partial fractions** corresponding to the given rational function.

We determine the constants A and B as follows:

Suppose $\frac{cx+d}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$ where $a \neq b$.

Then $\frac{cx+d}{(x-a)(x-b)} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)}$.

Therefore, $cx + d = A(x-b) + B(x-a)$

Substituting $x = a$ we obtain $A = \dfrac{ca+d}{a-b}$

Substituting $x = b$ we obtain $B = \dfrac{cb+d}{b-a}$.

Therefore, $\dfrac{cx+d}{(x-a)(x-b)} = \dfrac{ca+d}{(a-b)(x-a)} + \dfrac{cb+d}{(b-a)(x-b)}$

We may also obtain the constants by equating the coefficients on the left and right sides of $cx + d = A(x-b) + B(x-a)$.

x: $\quad c = A + B$

constant $\quad d = -bA - aB$

and solving these simultaneous equations for A and B

We give below by examples, the method of finding the partial fractions of rational functions that are proper fractions.

Case I: The denominator $D(x)$ is a product of distinct linear factors $(x-a_1),(x-a_2)....,(x-a_r)$.

Then $\dfrac{N(x)}{D(x)} = \dfrac{N(x)}{(x-a_1)(x-a_2)...(x-a_r)} = \dfrac{A_1}{(x-a_1)} + \dfrac{A_2}{(x-a_2)} + ... + \dfrac{A_r}{(x-a_r)}$

Example: $\dfrac{2x+1}{(x-1)(x-2)(x-3)}$

Let $\dfrac{2x+1}{(x-1)(x-2)(x-3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-3)}$.

Then $\dfrac{2x+1}{(x-1)(x-2)(x-3)} = \dfrac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$.

Therefore, $2x+1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$.

Substituting $x = 1$ we obtain 3 = A(-1)(-2). Therefore, $A = \dfrac{3}{2}$.

Substituting $x = 2$ we obtain 5 = B(1)(-1). Therefore, $B = -5$.

Substituting $x = 3$ we obtain 7 = C(2)(1). Therefore, $C = \dfrac{7}{2}$.

Thus $\dfrac{2x+1}{(x-1)(x-2)(x-3)} = \dfrac{3}{2(x-1)} - \dfrac{5}{(x-2)} + \dfrac{7}{2(x-3)}$

Case II: The denominator $D(x) = (x - a)^k$

Then $\dfrac{N(x)}{D(x)} = \dfrac{A_1}{(x-a)} + \dfrac{A_2}{(x-a)^2} + ... + \dfrac{A_k}{(x-a)^k}$.

Example: $\dfrac{x-1}{(x+3)^3}$

Let $\dfrac{x-1}{(x+3)^3} = \dfrac{A}{(x+3)} + \dfrac{B}{(x+3)^2} + \dfrac{C}{(x+3)^3}$

Then $x - 1 = A(x+3)^2 + B(x+3) + C$

Substituting x = -3 we obtain -4 = C

Equating the coefficient of x^2 we obtain 0 = A

Equating the coefficient of x we obtain 1 = 6A + B. Since A = 0 we obtain B = 1.

Thus $\dfrac{x-1}{(x+3)^3} = \dfrac{1}{(x+3)^2} - \dfrac{4}{(x+3)^3}$

Case III: The denominator $D(x)$ is a product of distinct irreducible quadratics $(x^2 + B_1 x + C_1), (x^2 + B_2 x + C_2), ..., (x^2 + B_k x + C_k)$.

Then $\dfrac{N(x)}{D(x)} = \dfrac{P_1 x + Q_1}{x^2 + B_1 x + C_1} + \dfrac{P_2 x + Q_2}{x^2 + B_2 x + C_2} + ... + \dfrac{P_k x + Q_k}{x^2 + B_k x + C_k}$

Example: $\dfrac{2x-1}{(x^2+1)(x^2+2)}$

Let $\dfrac{2x-1}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Dx+E}{x^2+2}$.

Then $2x - 1 = (Ax+B)(x^2+2) + (Dx+E)(x^2+1)$

Substituting x = 0 we obtain -1 = 2B + E ------------(i)

Equating the coefficient of x^3 we obtain 0 = A + D. i.e., D = -A -----------------(ii)

Equating the coefficient of x^2 we obtain 0 = B + E. Thus B = -E ---------------(iii)

Substituting this into (i) we obtain E = 1 and hence B = -1

Equating the coefficient of x we obtain 2 = 2A + D.

Substituting for D from (ii) we obtain A = 2 and therefore D = -2

Thus $\dfrac{2x-1}{(x^2+1)(x^2+2)} = \dfrac{2x-1}{x^2+1} + \dfrac{-2x+1}{x^2+2}$

Case IV: The denominator $D(x)$ = $(x^2 + bx + c)^k$ where $x^2 + bx + c$ is irreducible.

Then $\dfrac{N(x)}{D(x)} = \dfrac{P_1 x + Q_1}{x^2 + bx + c} + \dfrac{P_2 x + Q_2}{(x^2 + bx + c)^2} + ... + \dfrac{P_k x + Q_k}{(x^2 + bx + c)^k}$

Example: $\dfrac{x^3}{(x^2+2)^2}$

Let $\dfrac{x^3}{(x^2+2)^2} = \dfrac{Ax+B}{(x^2+2)} + \dfrac{Dx+E}{(x^2+2)^2}$.

Then $x^3 = (Ax+B)(x^2+2) + Dx + E$

Substituting $x = 0$ we obtain $0 = B + E$. Therefore, $B = -E$.

Equating the coefficient of x^3 we obtain $1 = A$.

Equating the coefficient of x^2 we obtain $0 = B$. Thus $E = 0$.

Equating the coefficient of x we obtain $0 = 2A + D$. Therefore, $D = -2$.

Therefore, $\dfrac{x^3}{(x^2+2)^2} = \dfrac{x}{(x^2+2)} - \dfrac{2x}{(x^2+2)^2}$

Case V: $D(x)$ is a combination of the above cases. Then, linear factors are handled as in cases I and II and quadratic factors are handled as in cases III and IV.

Example: $\dfrac{x^2-2x}{(x+1)(x-1)^2(x^2+1)}$

Let $\dfrac{x^2-2x}{(x+1)(x-1)^2(x^2+1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2} + \dfrac{Dx+E}{x^2+1}$.

Then

$x^2 - 2x = A(x-1)^2(x^2+1) + B(x+1)(x-1)(x^2+1) + C(x+1)(x^2+1) + (Dx+E)(x+1)(x-1)^2$

Substituting $x = 1$ we obtain $-1 = 4C$. Therefore, $C = -\dfrac{1}{4}$

Substituting $x = -1$ we obtain $3 = 8A$. Therefore, $A = \dfrac{3}{8}$

Equating the coefficient of x^4 we obtain $0 = A + B + D$. Therefore, $B + D = -\dfrac{3}{8}$ ------- (i)

Equating the coefficient of x^3 we obtain $0 = -2A + C - D + E$ --------(ii)

Substituting $x = 0$ we obtain $0 = A - B + C + E$ --------(iii)

Subtracting (ii) from (iii) we obtain

$0 = 3A - B + D$ --------(iv)

Adding (i) and (iv) we obtain $-\dfrac{3}{8} = 3A + 2D$. Thus $D = \dfrac{1}{2}(-\dfrac{3}{8} - \dfrac{9}{8}) = -\dfrac{3}{4}$.

Therefore $B = -\dfrac{3}{8} + \dfrac{3}{4} = \dfrac{3}{8}$.

Equating the coefficient of x^2 we obtain $1 = 2A + C - D - E$ --------(v)

Adding (ii) to (v) we obtain $1 = 2C - 2D$. Thus $C = \dfrac{1}{2}(1 - \dfrac{3}{2}) = -\dfrac{1}{4}$.

Therefore by (iii) we obtain $E = -A + B - C = -\dfrac{3}{8} + \dfrac{3}{8} + \dfrac{1}{4} = \dfrac{1}{4}$

Therefore $\dfrac{x^2-2x}{(x+1)(x-1)^2(x^2+1)} = \dfrac{\frac{3}{8}}{x+1} + \dfrac{\frac{3}{8}}{x-1} + \dfrac{-\frac{1}{4}}{(x-1)^2} + \dfrac{-\frac{3}{4}x+\frac{1}{4}}{x^2+1}$

$$= \frac{3}{8(x+1)} + \frac{3}{8(x-1)} - \frac{1}{4(x-1)^2} + \frac{-3x+1}{4(x^2+1)}$$

6. **Integration by Partial Fractions** (Ref 1: pg. 657, Ref 2: pg. 304-309)

We resolve the rational function $\dfrac{N(x)}{D(x)}$ into partial fractions when we wish to determine $\displaystyle\int \frac{N(x)}{D(x)}dx$ where $\dfrac{N(x)}{D(x)}$ is a proper fraction.

Example:

(1) $\displaystyle\int \frac{2x+1}{(x-1)(x-2)(x-3)}dx$

From the previous section we have

$$\frac{2x+1}{(x-1)(x-2)(x-3)} = \frac{3}{2(x-1)} - \frac{5}{(x-2)} + \frac{7}{2(x-3)}$$

Therefore,

$$\int \frac{2x+1}{(x-1)(x-2)(x-3)}dx = \int [\frac{3}{2(x-1)} - \frac{5}{(x-2)} + \frac{7}{2(x-3)}]dx$$

$$= \frac{3}{2}\int \frac{1}{(x-1)}dx - 5\int \frac{1}{(x-2)}dx + \frac{7}{2}\int \frac{1}{(x-3)}dx$$

$$= \frac{3}{2}\ln|x-1| - 5\ln|x-2| + \frac{7}{2}\ln|x-3| + C$$

(2) $\displaystyle\int \frac{x-1}{(x+3)^3}dx$

From the previous section we have $\dfrac{x-1}{(x+3)^3} = \dfrac{1}{(x+3)^2} - \dfrac{4}{(x+3)^3}$

Therefore,

$$\int \frac{x-1}{(x+3)^3}dx = \int [\frac{1}{(x+3)^2} - \frac{4}{(x+3)^3}]dx$$

$$= \int (x+3)^{-2}\,dx - 4\int (x+3)^{-3}\,dx$$

$$= -\frac{1}{(x+3)} + \frac{2}{(x+3)^2} + C$$

(3) $\displaystyle\int \frac{2x-1}{(x^2+1)(x^2+2)}\,dx$

From the previous section we have

$$\frac{2x-1}{(x^2+1)(x^2+2)} = \frac{2x-1}{x^2+1} + \frac{-2x+1}{x^2+2}$$

Therefore,

$$\int \frac{2x-1}{(x^2+1)(x^2+2)}\,dx = \int [\frac{2x-1}{x^2+1} + \frac{-2x+1}{x^2+2}]dx$$

$$= \int \frac{2x-1}{x^2+1}\,dx + \int \frac{-2x+1}{x^2+2}\,dx$$

$$= \int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx - \int \frac{2x}{x^2+2}\,dx + \int \frac{1}{x^2+(\sqrt{2})^2}dx$$

$$= \ln(x^2+1) - \tan^{-1} x - \ln(x^2+2) + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

(4) $\displaystyle\int \frac{x^2-2x}{(x+1)(x-1)^2(x^2+1)}\,dx$

From the previous section we have

$$\frac{x^2-2x}{(x+1)(x-1)^2(x^2+1)} = \frac{3}{8(x+1)} + \frac{3}{8(x-1)} - \frac{1}{4(x-1)^2} + \frac{-3x+1}{4(x^2+1)}$$

Therefore,

$$\int \frac{x^2-2x}{(x+1)(x-1)^2(x^2+1)}\,dx$$

$$= \int \frac{3}{8(x+1)}\,dx + \int \frac{3}{8(x-1)}\,dx - \int \frac{1}{4(x-1)^2}dx + \int \frac{-3x+1}{4(x^2+1)}\,dx$$

$$= \frac{3}{8}\int \frac{1}{x+1}\,dx + \frac{3}{8}\int \frac{1}{x-1}\,dx - \frac{1}{4}\int (x-1)^{-2}\,dx - \frac{3}{8}\int \frac{2x}{x^2+1}\,dx + \frac{1}{4}\int \frac{1}{x^2+1}\,dx$$

$$= \frac{3}{8}\ln|x+1| + \frac{3}{8}\ln|x-1| + \frac{1}{4(x-1)} - \frac{3}{8}\ln(x^2+1) + \frac{1}{4}\tan^{-1} x + C$$

7. **<u>The Definite Integral and the Area Under a Curve</u>** (Ref 1: pg. 635-636, Ref 2: pg. 206-209, 217-218, 257-260)

The Greek capital letter Σ denotes repeated addition. If f is a function defined on the integers, and if n and k are integers such that $n \geq k$, then

$$\sum_{j=k}^{n} f(j) = f(k) + f(k+1) + ... + f(n).$$

Area under a curve:

Suppose f is function such that $f(x) \geq 0$ for all x in the interval $[a, b]$. Then the graph of f lies on or above the x-axis.
Let $x_0, x_1, ..., x_n$ be points such that $a = x_0 < x_1 < ..... < x_n = b$.
Let the length of the subintervals $[x_0, x_1]$, $[x_1, x_2]$, …..,$[x_{n-1}, x_n]$ be denoted by $\Delta_1 x, \Delta_2 x, ..., \Delta_n x$ respectively. Then for $1 \leq k \leq n$, $\Delta_k x = x_k - x_{k-1}$.

Let R be the region bounded by the graph of the function, the x-axis, $x = a$ and $x = b$. This region may be divided into n strips by drawing vertical line segments $x = x_k$ from the x-axis up to the graph. If we denote the area of the k^{th} strip by $\Delta_k A$, then the total area of the region R is given by $A = \sum_{k=1}^{n} \Delta_k A$.

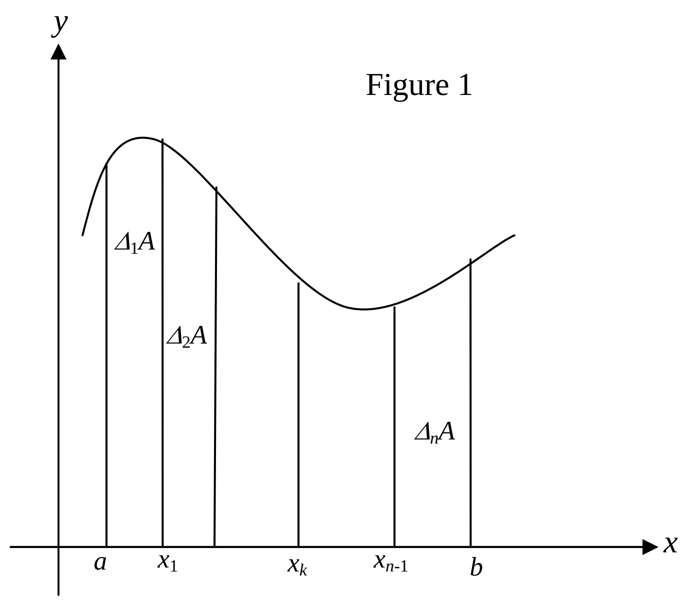
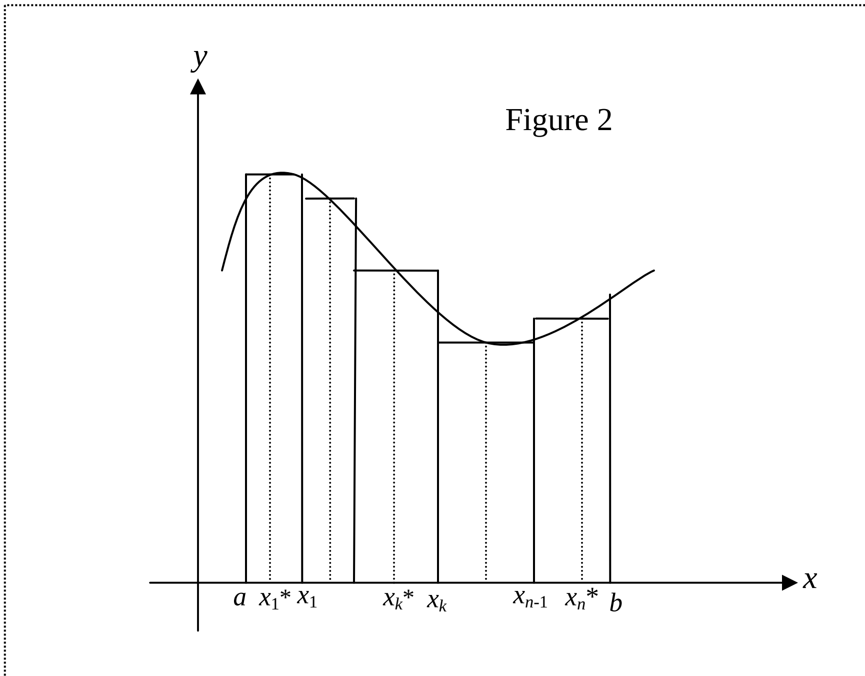


Figure 1

We may approximate the area $\Delta_k A$ by selecting a point x_k^* in the interval $[x_{k-1}, x_k]$ and computing the area of the rectangle with height $f(x_k^*)$ and width $\Delta_k x$ (Figure 2).



Then

$$\sum_{k=1}^{n} f(x_k^*)\Delta_k x = f(x_1^*)\Delta_1 x + f(x_2^*)\Delta_2 x + ... + f(x_n^*)\Delta x_n \text{ ----------------- (1)}$$

is an approximation of the total area A of the region R. We obtain better approximations by increasing the number of subintervals and reducing the lengths of the subintervals.
If the limit of this sum exists as the number of subintervals approaches infinity and the maximum length of the subintervals approaches zero, then this limit equals the area A under the curve and is called **the definite integral of *f* from *a* to *b*** and is denoted by $\int_a^b f(x)dx$.

In the notation $\int_a^b f(x)dx$, b is called the **upper limit** and a is called the **lower limit** of the definite integral.

For any function f (not necessarily non-negative), defined on the interval [a, b], sums of the form (1) above may be formed (without using the notion of area). If these sums tend to a finite limit as n, the number of subintervals tends to infinity, and the maximum of the lengths $\Delta_k x$ tends to 0, then the limit is denoted by $\int_a^b f(x)dx$ and is called the **definite integral of *f* on [*a, b*]**. If $\int_a^b f(x)dx$ exists, we say that f is **integrable** on [a, b].

<u>Result</u>:

If f is a function such that $f(x) \leq 0$ and integrable on [a, b], then the area A of the region bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by $A = -\int_a^b f(x)dx$.

<u>Theorem</u>:

If the function f is continuous on [a, b], then $\int_a^b f(x)dx$ exists.

<u>Result</u>:

If f is a function which is continuous on the interval [a, b], then $\int_a^b f(x)dx$ is equal to the area above the x-axis bounded by the graph of $y = f(x)$ from a to b **minus** the area below the x-axis bounded by the graph of $y = f(x)$ from a to b.

<u>Properties of the definite integral</u>

(1) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ where c is a constant.

(2) $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where $a < c < b$

(4) $\int_a^a f(x)dx = 0$

(5) $\int_b^a f(x)dx = -\int_a^b f(x)dx$

<u>The Fundamental Theorem of Calculus</u>

Let f be continuous on [a, b] and let $F(x) = \int f(x)dx$; i.e., F is an antiderivative of f. Then $\int_a^b f(x)dx = F(b) - F(a)$.

The fundamental theorem of calculus provides a simple way of computing the definite integral $\int_a^b f(x)dx$ when we can find an antiderivative of f. $F(b) - F(a)$ is often abbreviated as $F(x)]_a^b$.

Example:

(1) $\int_2^3 \frac{x^2+1}{x-1}dx = \int_2^3 (x+1+\frac{2}{x-1})dx = (\frac{x^2}{2}+x+2\ln|x-1|)]_2^3$

$$= (\frac{9}{2}+3+2\ln 2)-(\frac{4}{2}+2+2\ln 1)$$

$$= 3.5 + 2\ln 2$$

(2) $\int_0^1 \frac{1}{x^2+3}dx = \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}]_0^1 = \frac{1}{\sqrt{3}}(\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1} 0) = \frac{1}{\sqrt{3}}(\frac{\pi}{6}) = \frac{\pi}{6\sqrt{3}}$

(3) Find the area bounded by the graph of $y = -x$ and the x-axis between $x = -2$ and $x = 2$.

$$\text{Area} = \int_{-2}^{0}(-x)dx - \int_0^2 (-x)dx$$

$$= (\frac{-x^2}{2})]_{-2}^0 - (\frac{-x^2}{2})]_0^2$$

$$= [0 - \frac{-(-2)^2}{2}] - [\frac{-(2)^2}{2} - 0]$$

$$= 2 + 2 = 4$$

Note: $\int_{-2}^{2} -x\,dx = \frac{-x^2}{2}]_{-2}^2 = \frac{-(2)^2}{2} - \frac{-(-2)^2}{2} = 0$

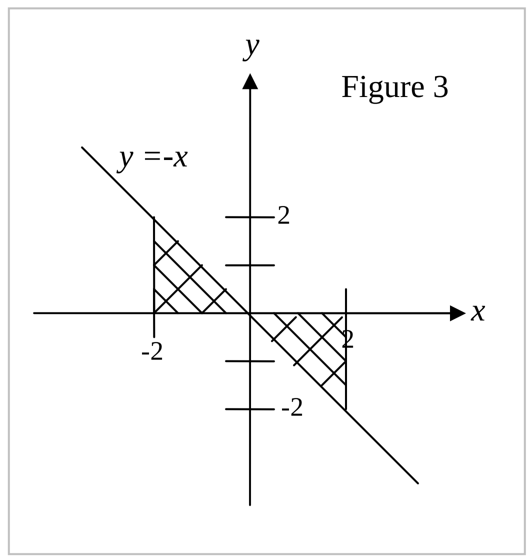


Figure 3

Area between two curves

Suppose f and g are continuous functions such that $g(x) \le f(x)$ for $a \le x \le b$. Then the curve $y = f(x)$ lies on or above the curve $y = g(x)$ between $x = a$ and $x = b$. The area A of the region between the two curves lying between $x = a$ and $x = b$ is given by

$$A = \int_a^b (f(x) - g(x))dx$$

Example:
Find the area between the graphs of $y = x^2$ and $y = -x$ in the interval [-1, 1]

Area between the two curves:

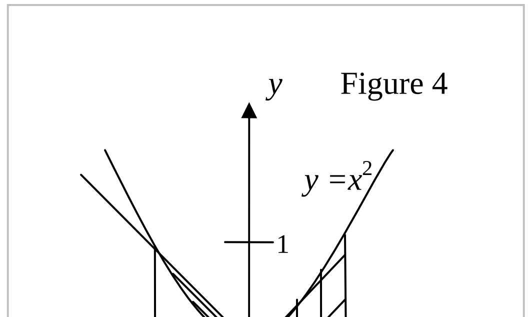


Figure 4

$$A = \int_{-1}^{0}(-x - x^2)dx + \int_{0}^{1}(x^2 - (-x))dx$$

$$= (\frac{-x^2}{2} - \frac{x^3}{3})]_{-1}^{0} + (\frac{x^3}{3} + \frac{x^2}{2})]_{0}^{1}$$

$$= [0 - (-\frac{1}{2} + \frac{1}{3})] + [\frac{1}{3} + \frac{1}{2}]$$

$$= 1$$

<u>Change of variable in a definite integral</u>

In computing a definite integral using the fundamental theorem of calculus, an antiderivative $\int f(x)dx$ is required. In section 3 we saw that sometimes it helps to substitute a new variable u to find $\int f(x)dx$. When a substitution is done to find $\int_a^b f(x)dx$, the limits of integration must be replaced by the corresponding values of u.

<u>Example</u>:

(1) $\int_0^{\frac{\pi}{4}} \sin^3 x \cos x dx$

Let $u = \sin x$. Then $\frac{du}{dx} = \cos x$.

When $x = 0$, $u = \sin 0 = 0$, and when $x = \frac{\pi}{4}$, $u = \frac{1}{\sqrt{2}}$.

Therefore, $\int_0^{\frac{\pi}{4}} \sin^3 x \cos x dx = \int_0^{\frac{1}{\sqrt{2}}} u^3 du = \frac{u^4}{4}]_0^{\frac{1}{\sqrt{2}}} = \frac{1}{16}$