<u>Differentiation</u>

References: (1) Business Mathematics by Qazi Zameeruddin, V. K. Khanna and S.K. Bhambri
(2) Schaum's Outlines, Calculus by Frank Ayres, Jr. and Elliot Mendelson (4th Edition)

1. <u>Functions</u>

<u>Example</u>:

Consider the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$.

In this formula, the value of π remains unchanged in the varying cases to which the formula applies. Thus π represents **a constant**. $\frac{4}{3}$ is also a constant in the above formula. For different spheres, r and V take different values; with ***V* depending on the value of *r***. V and r represent **variables**. Since the value of V depends on the value of r, V is called the **dependent variable**, and r is called the **independent variable**.

When two quantities are related as in the above example, the dependent variable is said to be a **function** of the independent variable.

Thus, in the above example, the volume of a sphere is a function of the radius of the sphere.

<u>Definition 1</u>:
A **function** is a rule that assigns to each value of the independent variable a **unique value** of the dependent variable.

The set of values that the independent variable takes is called **the domain** of the function. If f denotes the function and x the independent variable, $f(x)$ denotes the **value of *f* at *x*.** The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain. If y is a function of the independent variable x, we write $y = f(x)$ to indicate the dependence of y on x.

<u>Definition 2</u>:
The **composite function** $f_o g$ of the functions g and f is the function defined by $(f_o g)(x) = f(g(x))$.

2. <u>Limits and Continuity</u> [Ref 1: pg. 545-548, 551-552; Ref 2: pg. 61-63, 71 -73]

The concept of limit which is basic to the study of calculus, helps us to describe the behaviour of a function f when the independent variable x takes values very close to a particular value a.

Example:

Consider the function $f(x) = \dfrac{2x^2 - 2}{x - 1}$, $(x \neq 1)$. Let us consider what happens to the function as x approaches the value 1.

$x < 1$	$f(x)$
0.5	3
0.9	3.8
0.99	3.98
0.999	3.998
0.9999	3.9998
0.99999	3.99998
0.999999	3.999998

$x > 1$	$f(x)$
1.5	5
1.1	4.2
1.01	4.02
1.001	4.002
1.0001	4.0002
1.00001	4.00002
1.000001	4.000002

We see that as x approaches 1 (from values less than 1 as well as from values greater than 1), $f(x)$ approaches 4. We say that the **limit** of the function $f(x) = \dfrac{2x^2 - 2}{x - 1}$ as x approaches 1 is 4 and we write this as $\lim\limits_{x \to 1} \dfrac{2x^2 - 2}{x - 1} = 4$.

We note that the limit of the function as x approaches 1 exists, although the function itself is not defined at the point $x = 1$.

Now consider the following 2 tables.

$x < 1$	$1 - x$	$f(x)$	$4 - f(x)$
0.9	0.1	3.8	0.2
0.99	0.01	3.98	0.02
0.999	0.001	3.998	0.002
0.9999	0.0001	3.9998	0.0002
0.99999	0.00001	3.99998	0.00002
0.999999	0.000001	3.999998	0.000002

$x > 1$	$x - 1$	$f(x)$	$f(x) - 4$
1.1	0.1	4.2	0.2
1.01	0.01	4.02	0.02
1.001	0.001	4.002	0.002
1.0001	0.0001	4.0002	0.0002
1.00001	0.00001	4.00002	0.00002
1.000001	0.000001	4.000002	0.000002

We see from the 2nd and 4th columns of these tables that the closer x is to 1, the closer $f(x)$ is to 4. For example, if x differs from 1 by 0.0001, then $f(x)$ differs from 4 by

0.0002, and if x differs from 1 by 0.000001, then $f(x)$ differs from 4 by 0.000002. We see that we can make $f(x)$ as close to 4 as we please by taking x close enough to 1; i.e., we can make the absolute value of the difference between $f(x)$ and 4 as small as we please by making the difference between x and 1 small enough. Another way of saying this is: given any positive number ε, there is a positive number δ such that $|f(x)-4|<\varepsilon$ whenever $0<|x-1|<\delta$.

Let f be a function of x. We say that the limit of $f(x)$ as x approaches a is L if $f(x)$ gets arbitrarily close to L as x gets arbitrarily close to a, and we write this as $\lim_{x\to a} f(x)=L$.

We say that the limit of $f(x)$ as x approaches a from the left is L if $f(x)$ gets arbitrarily close to L as x gets arbitrarily close to a from values less than a, and we write this as $\lim_{x\to a^-} f(x)=L$.

We say that the limit of $f(x)$ as x approaches a from the right is L if $f(x)$ gets arbitrarily close to L as x gets arbitrarily close to a from values greater than a, and we write this as $\lim_{x\to a^+} f(x)=L$.

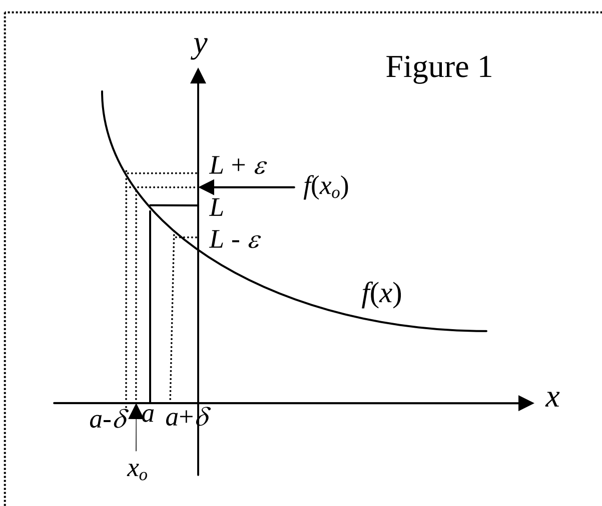
<u>Definition 3</u>:

(1) $\lim_{x\to a} f(x)=L$ if and only if given any positive number ε, there exists a corresponding positive number δ such that $|f(x)-L|<\varepsilon$ whenever $0<|x-a|<\delta$.

(2) $\lim_{x\to a^-} f(x)=L$ if and only if given any positive number ε, there exists a corresponding positive number δ such that $|f(x)-L|<\varepsilon$ whenever $a-\delta<x<a$.

(3) $\lim_{x\to a^+} f(x)=L$ if and only if given any positive number ε, there exists a corresponding positive number δ such that $|f(x)-L|<\varepsilon$ whenever $a<x<a+\delta$.

The idea of definition 3(1) is that if $\lim_{x\to a} f(x)=L$, then when $\varepsilon > 0$ is selected, $\delta > 0$ can be found, such that for any point x_0 in the interval $(a - \delta, a + \delta)$ such that $x_0 \neq a$, $f(x_0)$ lies in the interval $(L - \varepsilon, L + \varepsilon)$. (Figure 1)



Theorems:

(1) $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

(2) If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$, then $L_1 = L_2$ (i.e., if the limit exists, it is unique).

Theorems on Limits:

(1) If $f(x) = c$, a constant, then $\lim_{x\to a} f(x) = c$.

(2) Suppose $\lim_{x\to a} f(x) = A$, $\lim_{x\to a} g(x) = B$ and c is a constant. Then

(a) $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) = cA$.

(b) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = A \pm B$.

(c) $\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x).\lim_{x\to a} g(x) = A.B$.

(d) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{A}{B}$, if $B \neq 0$.

(e) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{A}$ provided $\sqrt[n]{A}$ is defined.

Example:

(1) $$\lim_{x\to 1} \frac{2(x^2-1)}{x-1} = \lim_{x\to 1} \frac{2(x-1)(x+1)}{(x-1)}$$
$$= \lim_{x\to 1} 2(x+1) = 2[\lim_{x\to 1} x + \lim_{x\to 1} 1] = 2(1+1) = 4$$

by applying 2(a), 2(b) and (1) of the theorems on limits.

(2) $$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)}$$
$$= \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{\lim_{x\to 4} 1}{\sqrt{\lim_{x\to 4} x} + \lim_{x\to 4} 2} = \frac{1}{4}$$

by applying 2(d), (1), 2(b) and 2(e) of the theorems on limits.

Definition 4:

A function f is said to be **continuous at the point x_o** if the following three conditions hold:

(1) $f(x_o)$ is defined;

(2) $\lim_{x\to x_o} f(x)$ exists;

(3) $\lim_{x\to x_o} f(x) = f(x_o)$.

If one or more of the above conditions fail, we say that the function is **discontinuous at x_o.**

We say that a function f is **continuous on a set *A*** if it is continuous at every point of A.

In the above example where $f(x) = \dfrac{2x^2-2}{x-1}$, ($x \neq 1$), the function is discontinuous at $x = 1$, since $f(1)$ is not defined.

Let us redefine the function as follows:

$$g(x) = \begin{cases} \dfrac{2x^2-2}{x-1} & x \neq 1 \\ 4 & x = 1 \end{cases}$$

Then g is continuous at $x = 1$.

Discontinuities of functions which may be removed by extending the function (as above) are called **removable discontinuities**.

<u>Infinity</u>

Let f be a function defined on some interval (a, ∞). We say that the limit of $f(x)$ as x approaches infinity is L, if the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large. We denote this by $\lim\limits_{x\to\infty} f(x) = L$.

Let f be a function defined on some interval (-∞, a). We say that the limit of $f(x)$ as x approaches negative infinity is L, if the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently small. We denote this by $\lim\limits_{x\to-\infty} f(x) = L$.

<u>Definition 5</u>:

(1) Let f be a function defined on some interval (a, ∞). Then $\lim\limits_{x\to\infty} f(x) = L$ if and only if given any real number $\varepsilon > 0$ there is a corresponding real number R such that $|f(x) - L| < \varepsilon$ whenever $x > R$.

(2) Let f be a function defined on some interval (-∞, a). Then $\lim\limits_{x\to-\infty} f(x) = L$ if and only if given any real number $\varepsilon > 0$ there is a corresponding real number R such that $|f(x) - L| < \varepsilon$ whenever $x < R$.

<u>Note</u>: **The Theorems on Limits also hold for limits at infinity; i.e., $x \to a$ in the theorems may be replaced by $x \to \infty$ as well as by $x \to -\infty$.**

Theorem:

(1) For r a positive rational number $\lim\limits_{x\to\infty}\frac{1}{x^r}=0$

(2) For r a positive rational number $\lim\limits_{x\to-\infty}\frac{1}{x^r}=0$ if x^r is defined for all x.

Example:

(1) $\lim\limits_{x\to-\infty}\frac{2x-x^2}{x^3}=\lim\limits_{x\to-\infty}(\frac{2}{x^2}-\frac{1}{x})=\lim\limits_{x\to-\infty}\frac{2}{x^2}-\lim\limits_{x\to-\infty}\frac{1}{x}=0+0=0$

by applying 2(b) of the theorems on limits and the above theorem.

(2) $\lim\limits_{x\to\infty}[\frac{3x^3-2x}{4x^3+x^2+1}]=\lim\limits_{x\to\infty}[\frac{3-\frac{2}{x^2}}{4+\frac{1}{x}+\frac{1}{x^3}}]=\frac{\lim\limits_{x\to\infty}3-\lim\limits_{x\to\infty}\frac{2}{x^2}}{\lim\limits_{x\to\infty}4+\lim\limits_{x\to\infty}\frac{1}{x}+\lim\limits_{x\to\infty}\frac{1}{x^3}}=\frac{3-0}{4+0+0}=\frac{3}{4}$

by applying the theorems on limits and the above theorem.

Let f be a function defined on both sides of a point a. We say that the limit of $f(x)$ as x approaches a is infinity, if the value of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a ($x\neq a$). We denote this by $\lim\limits_{x\to a}f(x)=\infty$.

Let f be a function defined on both sides of a point a. We say that the limit of $f(x)$ as x approaches a is negative infinity, if the value of $f(x)$ can be made arbitrarily small by taking x sufficiently close to a ($x\neq a$). We denote this by $\lim\limits_{x\to a}f(x)=-\infty$

Definition 6:

(1) Let f be a function defined on an open interval containing a, except possibly at a itself. Then $\lim\limits_{x\to a}f(x)=\infty$ if and only if for every positive number M there exists a corresponding positive number δ such that $f(x) > M$ whenever $0<|x-a|<\delta$.

(2) Let f be a function defined on an open interval containing a, except possibly at a itself. Then $\lim\limits_{x\to a}f(x)=-\infty$ if and only if for every negative number N there exists a corresponding positive number δ such that $f(x) < N$ whenever $0<|x-a|<\delta$.

(3) Let f be a function defined on an open interval (a, ∞). Then $\lim\limits_{x\to\infty}f(x)=\infty$ if and only if for every positive number M there exists a corresponding positive number R such that $f(x) > M$ whenever $x > R$.

Note: The definitions (1) and (2) can be extended to one-sided limits in the obvious way. Definitions similar to (3) can be obtained for $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

Theorems:

(1) If n is a positive even integer, then $\lim_{x\to a}\dfrac{1}{(x-a)^n} = \infty$

(2) If n is a positive odd integer, then $\lim_{x\to a^+}\dfrac{1}{(x-a)^n} = \infty$ and $\lim_{x\to a^-}\dfrac{1}{(x-a)^n} = -\infty$

Example:

(1) $\lim_{x\to 1^+}\dfrac{2(x+1)}{x-1} = +\infty$ since the denominator approaches 0 from the positive side while the numerator approaches the constant value 4 as x approaches 1 from the right.

(2) $\lim_{x\to 1^-}\dfrac{2(x+1)}{x-1} = -\infty$ since the denominator approaches 0 from the negative side while the numerator approaches the constant value 4 as x approaches 1 from the left.

3. **The Average Rate of Change** [Ref 2: pg. 79]

Definition 7:

Let y be a function of x; say $y = f(x)$. Suppose Δx represents a small change in the value of x from x_0 to $x_0 + \Delta x$, and suppose Δy represents the corresponding change in the value of y. Then $\Delta y = f(x_0 + \Delta x) - f(x_0)$.

The ratio $\dfrac{\Delta y}{\Delta x} = \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$ is called the **average rate of change** of y per unit change in x on the interval $[x_0, x_0 + \Delta x]$.

Example:

Suppose that the cost (in Rupees) of producing x units of a certain item is given by $C(x) = 10x + \dfrac{20}{x}$ ($x > 0$). Then the average rate of change of C with respect to x as x changes from 4 to 5 is $\dfrac{C(5) - C(4)}{5-4} = \dfrac{(50+4)-(40+5)}{5-4} = 9$. i.e., Rs. 9 per unit on the interval [4, 5].

4. **The Derivative** [Ref 1: pg. 553 – 554; Ref 2: pg. 79-80]

Definition 8:

Suppose $y = f(x)$ and x_0 is in the domain of f. Then, $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ is called **the instantaneous rate of change of f at x_0,** provided this limit exists finitely.

This limit is also called the **derivative of f at x_0** and is denoted by $f'(x_0)$.

The value of the derivative of a function f at an arbitrary point x in the domain of the function is given by $\lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$.

This is denoted by any one of the following expressions: y', $f'(x)$, f', $\dfrac{dy}{dx}$, $\dfrac{df(x)}{dx}$, $\dfrac{d}{dx} f$, $D_x y$.

A function f is said to be **differentiable** at the point x_0 provided the derivative of f exists at x_0.

A function f is said to be differentiable on a set, if f is differentiable at every point in the set.

The process of finding the derivative of a function is called **differentiation**.

Example:

Consider $f(x) = 3x^2 + 2x$

Then

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{[3(x + \Delta x)^2 + 2(x + \Delta x)] - [3x^2 + 2x]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[3x^2 + 6x\Delta x + 3\Delta x^2 + 2x + 2\Delta x] - [3x^2 + 2x]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{6x\Delta x + 3\Delta x^2 + 2\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} 6x + 3\Delta x + 2 \\
&= 6x + 2
\end{aligned}$$

i.e., $f'(x) = 6x + 2$

5. **Differentiation Formulas** [Ref 1: pg. 556-557, 563; Ref 2: pg. 86-87]

There are several rules for finding the derivative without using the definition directly. These formulas greatly simplify the task of differentiation.

(1) If f is a constant function $f(x) = c$, then $f'(x) = 0$.

Suppose f and g are differentiable functions, c is a constant and n is a real number. Then,

(2) $\frac{d}{dx}(cf) = c\frac{df}{dx}$

(3) $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$ (The Sum Rule)

(4) $\frac{d}{dx}(f-g) = \frac{df}{dx} - \frac{dg}{dx}$ (The Difference Rule)

(5) $\frac{d}{dx}(f.g) = f\frac{dg}{dx} + g\frac{df}{dx}$ (The Product Rule)

(6) $\frac{d}{dx}(\frac{f}{g}) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$ provided that $g \neq 0$ (The Quotient Rule)

(7) $\frac{d}{dx}(x^n) = nx^{n-1}$ (The Power Rule)

(8) $\frac{d}{dx}(f(g(x)) = f'(g(x)).g'(x)$ (The Chain Rule)

An alternate formulation of the chain rule: Suppose $y = f(u)$ and $u = g(x)$. Then $y = f(g(x))$ is the composite of the functions g and f and $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

<u>Example</u>:

(1) Let $y = 3x^3 + x^2 - 4$.

Then $y' = \frac{d}{dx}(3x^3) + \frac{d}{dx}(x^2) - \frac{d}{dx}(4)$ applying (3) and (4) above.

$= 3\frac{d}{dx}(x^3) + \frac{d}{dx}(x^2) - \frac{d}{dx}(4)$ applying (2)

$= 9x^2 + 2x$ applying (1) and (7)

(2) Let $y = (5x+2)(3x^{\frac{1}{3}} + 1)$

Then $y' = (5x+2)\frac{d}{dx}(3x^{\frac{1}{3}} + 1) + (3x^{\frac{1}{3}} + 1)\frac{d}{dx}(5x+2)$ applying (5)

$= (5x+2)(x^{-\frac{2}{3}}) + (3x^{\frac{1}{3}} + 1)(5)$ applying (1), (2), (3), (7)

(3) Let $y = \frac{x^3 + 3x}{\sqrt{x}}$

Then $y' = \dfrac{\sqrt{x}\,\dfrac{d}{dx}(x^3+3x) - (x^3+3x)\dfrac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}$ applying (6)

$= \dfrac{\sqrt{x}(3x^2+3) - (x^3+3x)(\frac{1}{2}x^{-\frac{1}{2}})}{x}$ applying (2), (3), (7)

(4) Let $y = \sqrt{(5x^4 - 3x^2)}$

Then $y' = \dfrac{1}{2}(5x^4 - 3x^2)^{-\frac{1}{2}}(20x^3 - 6x)$ applying (8), (2), (4), (7)

6. <u>Standard Derivatives</u>

[Ref 1: pg. 558-561, 567-568; Ref 2: pg. 153, 155-156, 166- 169, 225, 234-235, 237]

In the following table we give some standard derivatives

I. Power Functions:	$\dfrac{d}{dx}(x^n) = nx^{n-1}, \quad n \in R$
II. Exponential Functions:	$\dfrac{d}{dx}(a^x) = a^x \ln a \ \left(a > 0,\ a \neq 1\right)$ $\dfrac{d}{dx}(e^x) = e^x$
III. Logarithmic Functions:	$\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a} \ (x > 0)$ $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x} \ (x > 0)$
IV. Trigonometric Functions:	$\dfrac{d}{dx}(\sin x) = \cos x$ $\dfrac{d}{dx}(\cos x) = -\sin x$ $\dfrac{d}{dx}(\tan x) = \sec^2 x \quad (x \neq \dfrac{\pi}{2} + \pi n,\ n = 0, \pm 1, \pm 2, \ldots)$ $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \quad (x \neq \pi n,\ n = 0, \pm 1, \pm 2, \ldots)$ $\dfrac{d}{dx}(\sec x) = \sec x \tan x \quad (x \neq \dfrac{\pi}{2} + \pi n,\ n = 0, \pm 1, \pm 2, \ldots)$

	$$\frac{d}{dx}(\operatorname{cosec} x) = -\cot x \operatorname{cosec} x \ \ (x \neq \pi n,\ \ n = 0,\pm1,\pm2,....)$$
V. Inverse Trigonometric Functions: $$\frac{d}{dx}(\sin^{-1}\frac{x}{a}) = \frac{1}{\sqrt{a^2 - x^2}}, \quad (\lvert x\rvert < \lvert a\rvert)$$ $$\frac{d}{dx}(\cos^{-1}\frac{x}{a}) = -\frac{1}{\sqrt{a^2 - x^2}}, \quad (\lvert x\rvert < \lvert a\rvert)$$ $$\frac{d}{dx}(\tan^{-1}\frac{x}{a}) = \frac{a}{x^2 + a^2}$$ $$\frac{d}{dx}(\cot^{-1}\frac{x}{a}) = -\frac{a}{x^2 + a^2}$$ $$\frac{d}{dx}(\sec^{-1}\frac{x}{a}) = \frac{a}{x\sqrt{x^2 - a^2}}, \quad (\lvert x\rvert > \lvert a\rvert)$$ $$\frac{d}{dx}(\operatorname{cosec}^{-1}\frac{x}{a}) = -\frac{a}{x\sqrt{x^2 - a^2}}, \quad (\lvert x\rvert > \lvert a\rvert)$$	

Example:

(1) Let $y = \dfrac{a^x}{\sqrt{\sin x}}$.

Then $$y' = \frac{\sqrt{\sin x}a^x \ln a - a^x \frac{1}{2}(\sin x)^{-\frac{1}{2}}(\cos x)}{(\sqrt{\sin x})^2} = \frac{a^x[2(\ln a)(\sin x) - \cos x]}{2(\sin x)^{\frac{3}{2}}}$$

(2) Let $y = \tan x.\tan^{-1}\dfrac{x}{3}$

Then $$y' = \sec^2 x.\tan^{-1}\frac{x}{3} + \tan x.\frac{3}{x^2 + 9}$$

(3) Let $y = e^{\frac{\sin 3x}{\sec x}}$.

Then $$y' = e^{\frac{\sin 3x}{\sec x}}[\frac{(\sec x)(3\cos 3x) - (\sin 3x)(\sec x \tan x)}{(\sec x)^2}].$$

(4) Let $y = \ln[\dfrac{\cos^2(3x^3) - e^{4x}}{\cot x^2}]$

Then

$$y' = \frac{1}{[\frac{\cos^2(3x^3) - e^{4x}}{\cot x^2}]}.[\frac{(\cot x^2)[-2\cos(3x^3)\sin(3x^3)9x^2 - 4e^{4x}] - [\cos^2(3x^3) - e^{4x}](-\operatorname{cosec}^2 x^2)2x}{(\cot x^2)^2}]$$

7. **Higher Order Derivatives** [Ref 1: pg. 577; Ref 2: pg. 89]

Let $y = f(x)$ be a differentiable function. Its derivative y' is also called the **first derivative** of f. If y' is differentiable, its derivative is called **the second derivative** of f. If the second derivative is differentiable, its derivative is called the **third derivative** of f and so on.

We use the following notations:

First derivative of $y = f(x)$: $y',\ f'(x),\ \dfrac{dy}{dx},\ D_x y$

Second derivative of $y = f(x)$: $y'',\ f''(x), \dfrac{d^2y}{dx^2}, D_x^2 y$

Third derivative of $y = f(x)$: $y''',\ f'''(x),\ \dfrac{d^3y}{dx^3},\ D_x^3 y$

n^{th} derivative of $y = f(x)$: $y^{(n)},\ f^{(n)}, \dfrac{d^n y}{dx^n},\ D_x^n y$

Example:

(1) Suppose $y = \cos(3x+1)$.

Then $y' = -3\sin(3x+1)$

$y'' = -3^2\cos(3x+1)$

$y''' = 3^3\sin(3x+1)$

$$y^{(n)} = 3^n\cos[(3x+1)+\frac{n\pi}{2}] \qquad n \ge 1$$

(2) Suppose $y = \dfrac{1}{(x-1)^2}$

Then $y' = -2(x-1)^{-3}$

$y'' = (-2)(-3)(x-1)^{-4}$

$y''' = (-2)(-3)(-4)(x-1)^{-5}$

$y^{(n)} = (-1)^n(n+1)!(x-1)^{-(n+2)} \qquad n \ge 1$

8. **Applications**

[Ref 1: pg. 596-597, 606-610, Ref 2: pg. 102, 108, 110, 115-117, 129-130, 175-176]

Many physical phenomena (growth of plants, population growth, radio active decay etc.) involve changes in quantities with respect to other quantities. The average rate of change shows the change in the dependent variable per unit change of the independent variable in a given interval. If we consider the velocity of a vehicle, this usually varies with time. The instantaneous rate of change of the position function with respect to time would give the velocity at a given instance t.

Rectilinear motion -Velocity and Acceleration [Ref 2: pg. 175-176]

Let $s(t)$ denote the **position function** of an object moving along a straight line from a certain point. Then the average velocity from $t = t_o$ to $t = t_o + \Delta t$ is $\frac{s(t_o + \Delta t) - s(t_o)}{\Delta t}$.

The **instantaneous velocity** (or velocity) $v(t_o)$ of the object at time t_o is given by

$v(t_o) = \lim_{\Delta t \to 0} \frac{s(t_o + \Delta t) - s(t_o)}{\Delta t} = s'(t_o)$.

Therefore, if $s(t)$ denotes the position function of an object moving in a straight line from a certain initial point, then the **velocity function** $\boldsymbol{v(t)}$ of the object at time t is given by $v(t) = s'(t)$.

If the velocity is negative, this means that object is moving in the direction of decreasing s.

The derivative of the velocity function $v(t)$ is called the **acceleration function** and is denoted by $a(t)$. $a(t) = v'(t) = s''(t)$.

Example:

The distance s (in meters) an object moves in t seconds is given by $s(t) = 120t - 16t^2$.

Let us find the following:

(i) The velocity after 3 seconds
(ii) The acceleration after 3 seconds
(iii) The time the velocity is zero
(iv) The distance the body has travelled before coming to rest
(v) The velocity after 5 seconds, and the significance of its sign.

(i) The velocity after 3 seconds is $v(3) = s'(3)$
$s'(t) = 120 - 32t$.
Therefore, $s'(3) = [120 - (32)(3)]\ ms^{-1} = 24\ ms^{-1}$

(ii) The acceleration after 3 seconds is $a(3) = s''(3)$.
$s''(t) = -32$. Thus $s''(3) = -32\ ms^{-2}$

(iii) $v(t) = s'(t) = 120 - 32t = 0$ when $t = \frac{120}{32}\sec = \frac{15}{4}\sec = 3\frac{3}{4}\sec$.

(iv) When $t = 3\frac{3}{4}\sec$, $s(t) = [120(\frac{15}{4}) - 16(\frac{225}{16})]m = [450 - 225]m = 225m$.
i.e., The distance the body has travelled before coming to rest is $225m$.

(v) The velocity after 5sec is $v(5) = [120 - 32(5)]\ ms^{-1} = [120 - 160]\ ms^{-1} = -40\ ms^{-1}$.
This shows that the object is moving in a direction opposite to its initial direction after 5sec.

<u>Tangent Line to a curve</u> [Ref 1: pg. 596-597; Ref 2: pg. 102]

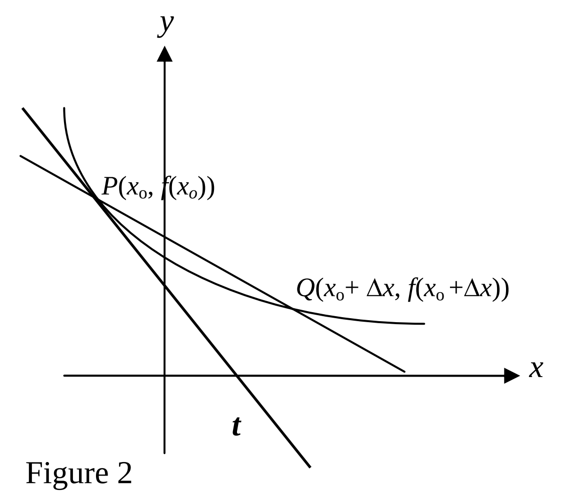


Figure 2

The tangent line to a curve at a point P is the line that touches the curve at point P.

To find the tangent line t to the graph of a function f at a point $P(x_o, f(x_o))$ we need to find the slope m of t. Since we only have one point P on t, we compute an approximation to m by selecting a point $Q(x_o+ \Delta x, f(x_o +\Delta x))$ close to P on the curve and computing the slope m_{PQ} of the secant line PQ (Figure 2). As the point Q moves towards the point P along the curve (Figure 3), if the slope m_{PQ} of the secant line PQ approaches the value m, then we define the tangent line t to the curve at the point P to be the line through P with slope m.

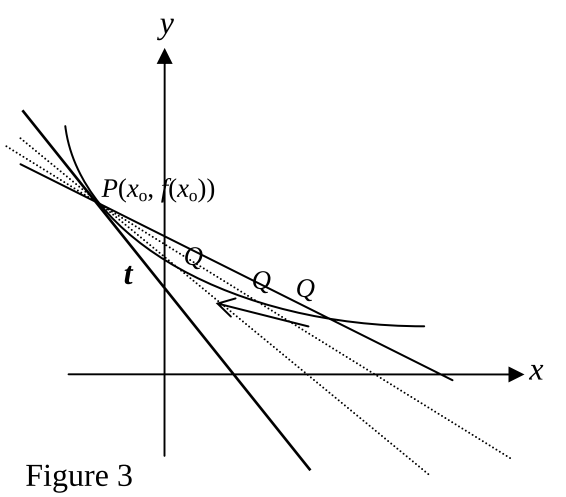


Figure 3

In the notation of limits we have

$$m = \lim_{Q\to P} m_{PQ} = \lim_{\Delta x\to 0} \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x}$$

$$= f'(x_o).$$

<u>Example</u>:

Let $f(x) = \dfrac{x}{1-x}$.

Then $f'(x) = \dfrac{(1-x).1 - x.(-1)}{(1-x)^2} = \dfrac{1}{(1-x)^2}$.

Therefore, $f'(3) = \dfrac{1}{4}$.

When $x = 3$, $f(3) = -\dfrac{3}{2}$.

Therefore, the tangent line to the curve of $f(x) = \dfrac{x}{1-x}$ at the point $(3, -\dfrac{3}{2})$ is given by $y - (-\dfrac{3}{2}) = \dfrac{1}{4}(x-3)$ or $y = \dfrac{1}{4}x - \dfrac{9}{4}$.

Curve Sketching- Maxima and Minima

[Ref 1: pg. 606-610; Ref 2: pg. 108, 110, 115-117, 129-130]

Definition 9:

(1) A function f is said to be **increasing** on an interval if whenever $x < y$ on the interval, we have $f(x) < f(y)$.

(2) A function f is said to be **decreasing** on an interval if whenever $x < y$ on the interval, we have $f(x) > f(y)$.

(3) A function f is said to have a **relative minimum** at x_o if $f(x) \geq f(x_o)$ for all x in some open interval (a, b) containing x_o (on which f is defined).

(4) A function f is said to have a **relative maximum** at x_o if $f(x) \leq f(x_o)$ for all x in some open interval (a, b) containing x_o (on which f is defined).

(5) A **relative extremum** of f is either a relative maximum or a relative minimum of f.

(6) A number x_o in the domain of the function f is called a **critical number** of f if either $f'(x_o) = 0$ or $f'(x_o)$ is not defined.

(7) An **absolute maximum** of a function f on a set S occurs at a point x_o in S if $f(x) \leq f(x_o)$ for all x in S.

(8) An **absolute minimum** of a function f on a set S occurs at a point x_o in S if $f(x) \geq f(x_o)$ for all x in S.

(9) A **point of inflection** on a curve $y = f(x)$ is a point at which the concavity changes; i.e., the curve is concave upward on one side of the point and concave downward on the other side.

(10) A vertical line $x = x_o$ such that $f(x)$ approaches $+\infty$ or $-\infty$ as x approaches x_o from either the left or the right side, is called a **vertical asymptote** of the graph of f.

(11) A horizontal line $y = y_o$ is called a **horizontal asymptote** of the graph of f if either $\lim_{x\to-\infty} f(x) = y_o$ or $\lim_{x\to\infty} f(x) = y_o$

Theorem:

(1) If f' is positive on an interval, then f is increasing on the interval.

(2) If f' is negative on an interval, then f is decreasing on the interval.

(3) If f has a relative extremum at a point x_o at which $f'(x_o)$ is defined, then $f'(x_o) = 0$.

(4) If $f'(x_o) = 0$ and $f''(x_o)$ exists, then
(a) if $f''(x_o) < 0$, f has a relative maximum at x_o.
(b) if $f''(x_o) > 0$, f has a relative minimum at x_o.
(c) if $f''(x_o) = 0$, we cannot conclude anything about f at x_o.
(This is called the second derivative test)

(5) Suppose $f'(x_o) = 0$.

(d) If f' is positive in an open interval (a, x_o) immediately to the left of x_o and negative on an open interval (x_o, b) immediately to the right of x_o, then f has a relative maximum at x_o.

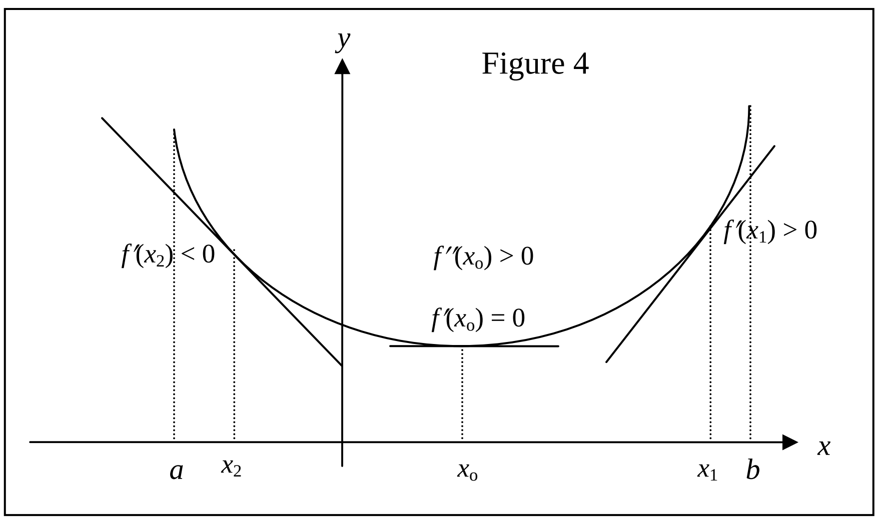
(e) If f' is negative in an open interval (a, x_o) immediately to the left of x_o and positive on an open interval (x_o, b) immediately to the right of x_o, then f has a relative minimum at x_o.

(f) If f' has the same sign in open intervals (a, x_o) and (x_o, b) immediately to the left and to the right of x_o, then f has neither a relative maximum nor a relative minimum at x_o.

(This is called the first derivative test)

(6) (a) If $f''(x) > 0$ for x in the open interval (a, b), then the graph of f is concave upward for $a < x < b$. (i.e., the graph is above the tangent line at x in (a, b)).

(b) If $f''(x) < 0$ for x in the open interval (a, b), then the graph of f is concave downward for $a < x < b$. (i.e., the graph is below the tangent line at x in (a, b)).



In Figure 4:

(i) The function is increasing on the interval (x_o, b) and $f'(x) > 0$ on this interval.

(ii) The function is decreasing on the interval (a, x_o) and $f'(x) < 0$ on this interval.

(iii) The function has a relative minimum at x_o, x_o is a critical value of f, $f'(x_o) = 0$ and $f''(x_o) > 0$.

(iv) $f''(x) > 0$ for x in the open interval (a, b), and hence the graph of f is concave upward for $a < x < b$.

(v) An absolute minimum on the interval $[a, b]$ occurs at x_o.

In Figure 5:

(i) The graph of the function is concave down on the interval (a, x_o).

(ii) The graph of the function is concave up on the interval (x_o, b).

(iii) $(x_o, f(x_o))$ is a point of inflection.

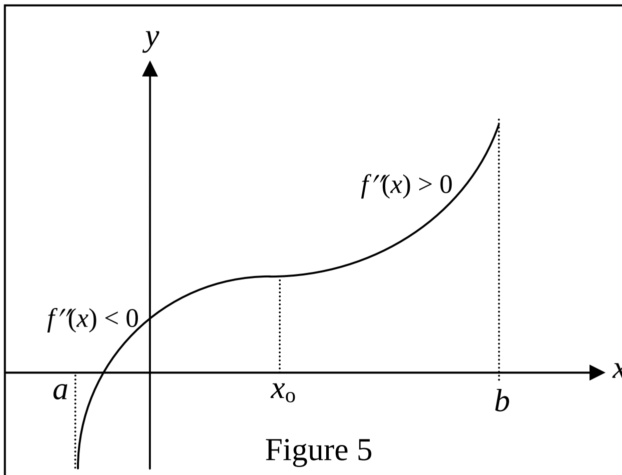


Figure 5

Example:
Let us consider the graph of the function $f(x) = 3x^4 - 16x^3 + 18x^2$ on the interval [-1, 4].

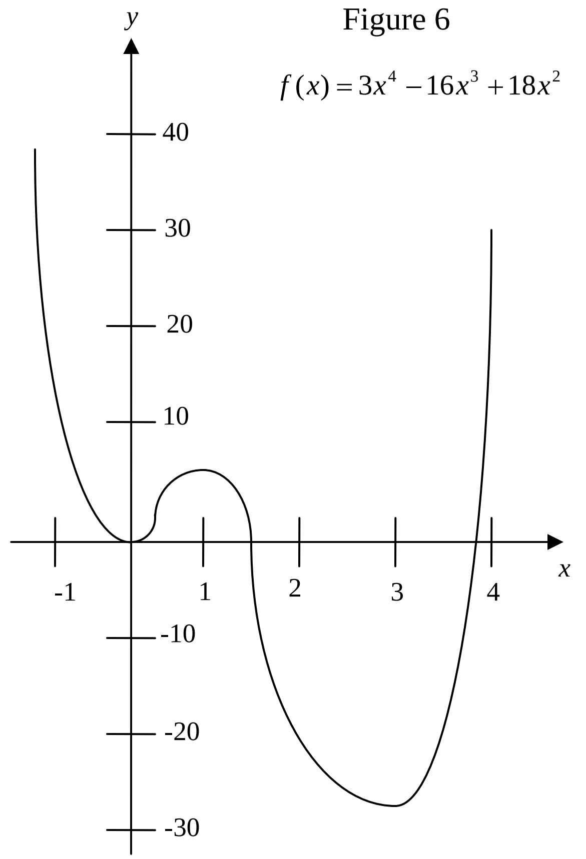


Figure 6

$$f'(x) = 12x^3 - 48x^2 + 36x$$
$$= 12x(x^2 - 4x + 3)$$
$$= 12x(x-3)(x-1)$$

$f'(x) = 0$ when $x = 0$, $x = 1$, $x = 3$.

$f'(x)$ is positive on (0, 1) and (3, 4), and therefore f is increasing on these intervals.

$f'(x)$ is negative on (-1, 0) and (1, 3) and therefore f is decreasing on these intervals.

$f(-1) = 37$, $f(0) = 0$, $f(1) = 5$, $f(3) = -27$ and $f(4) = 32$.

$f''(x) = 12(3x^2 - 8x + 3)$

$f''(0) = 36 > 0$. Therefore, (0, 0) is a relative minimum.

$f''(1) = -24 < 0$. Therefore, (1, 5) is a relative maximum.

$f''(3) = 72 > 0$. Therefore, (3, -27) is a relative minimum. It is also the absolute minimum on the interval [-1, 4]

Example:

Let $f(x) = \frac{1}{x}$ $(x \neq 0)$.

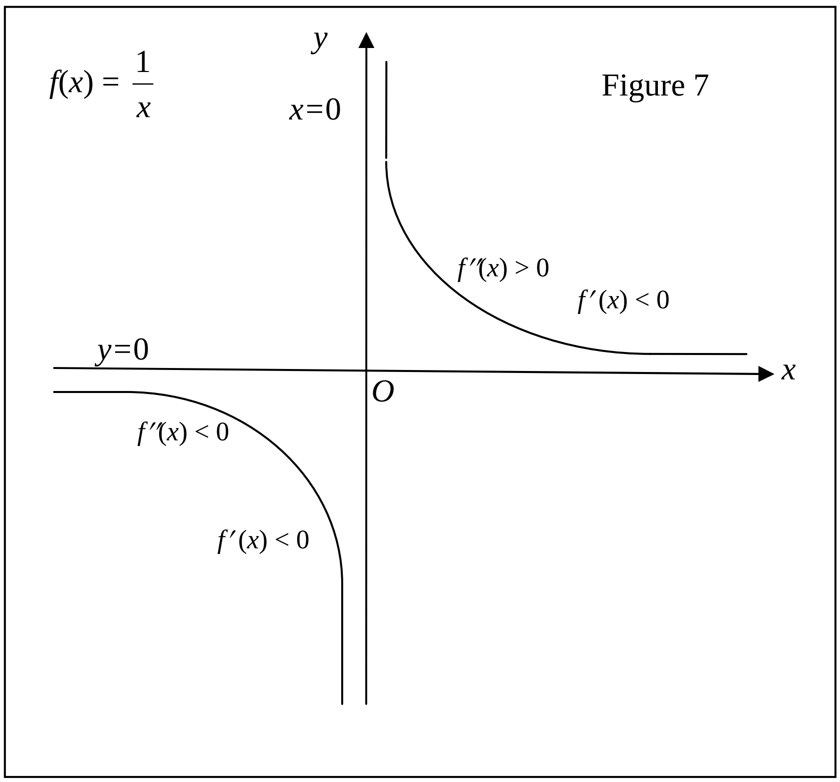
Then $f'(x) = -\frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$.

$f'(x) < 0$ for all x such that $x \neq 0$. Therefore, f is decreasing on $(-\infty, 0)$ and on $(0, \infty)$.

$f''(x) > 0$ when $x > 0$ and $f''(x) < 0$ when $x < 0$. Therefore, the graph is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

$\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to-\infty} \frac{1}{x} = 0$. Therefore, $y = 0$ is a horizontal asymptote of the graph of f.

$\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$. Therefore, $x = 0$ is a vertical asymptote of the graph of f.



9. **Indeterminate Forms and L'Hospital's Rule** [Ref 2: pg. 243 – 245]

Consider $\lim_{x\to 0} \frac{e^x - 1}{x}$. We cannot apply (d) of the theorem on limits (the limit of a quotient is the quotient of the limits) since the limit of the denominator is 0. Here we have that both the numerator and the denominator approach 0 as x approaches 0. A limit of this form (where both the numerator and denominator approach 0) may or may not exist.

Suppose we have a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$. We call a limit of this type an **indeterminate form of type** $\frac{0}{0}$.

Consider $\lim_{x \to \infty} \frac{\ln x}{x^2}$. Here both the numerator and the denominator approach ∞ as $x \to \infty$. A limit of this form may or may not exist.

Suppose we have a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$) as $x \to a$. We call a limit of this type an **indeterminate form of type** $\frac{\infty}{\infty}$.

A systematic method for evaluating indeterminate forms is **l'Hospital's Rule**.

<u>Theorem: L'Hospital's Rule</u>

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \text{ and } \lim_{x \to a} g(x) = \pm\infty .$$

(i.e., we have an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$**)**

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, if the limit on the right hand side exists (or is $+\infty$ or $-\infty$).

<u>Note:</u> **l'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity. i.e., $x \to a$ in l'Hospital's Rule can be replaced by any one of $x \to a^+$, $x \to a^-$, $x \to \infty$ or $x \to -\infty$.**

<u>Example</u>:

(1) $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$.

(2) $\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty$ by repeated application of l'Hospital's Rule.

Indeterminate Products

Consider $\lim_{x\to a} f(x).g(x)$ where $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty$. A limit of this kind is called an **indeterminate form of type 0.∞**. We convert this limit to an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by writing the product $f(x).g(x)$ as $f(x).g(x) = \frac{f(x)}{1/g(x)}$ or as $f(x).g(x) = \frac{g(x)}{1/f(x)}$ and then apply l'Hospital's Rule.

Example:

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x}.$$

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \quad \text{by applying l'Hospital's Rule}$$

$$= \lim_{x\to 0^+} (-x) = 0$$

Therefore, $\lim_{x\to 0^+} x\ln x = 0$.

Indeterminate Differences

If $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$, then $\lim_{x\to a}(f(x)-g(x))$ is called an **indeterminate form of type ∞ - ∞.** To find $\lim_{x\to a}(f(x)-g(x))$, we convert it into an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then apply l'Hospital's Rule.

Example:

$$\lim_{x\to \frac{\pi}{2}^+} (\sec x - \tan x) = \lim_{x\to \frac{\pi}{2}^+} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x\to \frac{\pi}{2}^+} \left(\frac{1-\sin x}{\cos x}\right).$$

$$\lim_{x\to \frac{\pi}{2}^+} \left(\frac{1-\sin x}{\cos x}\right) = \lim_{x\to \frac{\pi}{2}^+} \left(\frac{-\cos x}{-\sin x}\right) \quad \text{by applying l'Hospital's Rule}$$

$$= 0$$

Therefore, $\lim_{x\to \frac{\pi}{2}^+} (\sec x - \tan x) = 0$.

Indeterminate Powers

There are several indeterminate forms that arise from $\lim_{x\to a} [f(x)]^{g(x)}$.

(i) $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$ type 0^0

(ii) $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=0$ type ∞^0

(iii) $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^{∞}

Each of these can be treated by considering the natural logarithm of the function, which would give us an indeterminate product of type $0.\infty$.

<u>Example</u>:
Consider $\lim_{x \to 0^+} x^x$. This is of the type 0^0.
Let $y = x^x$. Then $\ln y = x\ln x$.
$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} x \ln x = 0$ (by the example above).
Therefore, $\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^0 = 1$.