Vectors

Reference: Business Mathematics by Qazi Zameeruddin, V. K. Khanna and S.K. Bhambri (Pages 762- 795)

1. Vectors and Scalars

There are quantities in the physical world that are characterized by just their magnitude, such as length, temperature, time, mass and speed. These quantities are called **scalars**. Apart from the units that they are measured in, these are just real numbers. We use ordinary letters such as *l*, *t*, *m*, *s* to represent them. Operations with scalars are carried out as in usual elementary algebra.

There are other quantities in the physical world that cannot be characterized by their magnitude alone, such as displacement, velocity, acceleration and force. These are characterized by both their magnitude and their direction. Such quantities are called **vectors**. We denote vectors by bold faced letters (e.g., ***A, b***) or letters with an arrow over them (e.g., $\vec{AB}, \vec{a}$).

Let *A*, *B* represent two distinct points in space. There exists exactly one straight line passing through both these points. That part of the line between *A* and *B* and including both the end points is called a **line segment**. If the end points *A* and *B* are given a definite order, then the line segment is said to be **directed** and its direction is denoted by an arrow as in Figure 1. *A* is then called the **initial point** and *B* the **terminal point** of the directed line segment.

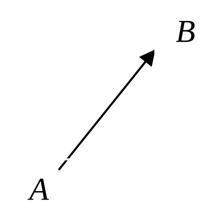


Figure 1

A directed line segment has both magnitude (distance between the two end points) and direction.

We define a vector as a directed line segment.

2. Equality of Vectors

Two vectors ***a*** and ***b*** are said to be **equal** if they have the same magnitude and direction. Thus in Figure 2, ***a*** = ***b*** = ***c***. Directed line segments which are equal in length and are in the same direction are called equivalent directed line segments. Thus any vector corresponds to a collection of equivalent directed line segments.

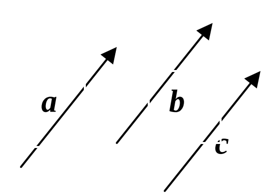


Figure 2

3. Magnitude of a Vector

The magnitude or modulus of the vector $\boldsymbol{a} = \overrightarrow{AB}$ is the length of the line segment joining the points A and B. This is a non-negative number and is denoted by the symbol $|\boldsymbol{a}|$ or a or $|\overrightarrow{AB}|$.

4. Special Vectors

Null Vector: A vector whose initial and terminal points coincide is called a **null vector** (or **zero vector)**. This vector has magnitude equal to zero and no specific direction. It is denoted by **0**.

Unit Vector: A vector with unit magnitude (length equal to 1) is called a **unit vector**.

If $\boldsymbol{A}$ is a vector which is non-zero, then $\boldsymbol{a} = \boldsymbol{A}/|\boldsymbol{A}|$ is a unit vector in the same direction as $\boldsymbol{A}$. Any vector $\boldsymbol{A}$ can be written as $\boldsymbol{A} = |\boldsymbol{A}|\,\boldsymbol{a}$ where $\boldsymbol{a}$ is a unit vector in the same direction as $\boldsymbol{A}$.

Negative of a Vector : If $\boldsymbol{a}$ is a vector, then $-\boldsymbol{a}$, which is called the negative of vector $\boldsymbol{a}$ is a vector which has the same magnitude as $\boldsymbol{a}$ but is in the opposite direction to $\boldsymbol{a}$.

Free Vector: A vector whose position in space is not fixed is called a **free vector** (or **sliding vector**).

Localised Vector : A vector whose position in space is fixed is called a **localized vector** (or **bound vector**)

Co-initial Vectors : Vectors with the same initial point in space are called **co-initial vectors** (or **concurrent vectors**)

5. Vector Addition and Subtraction

Addition of Vectors

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two vectors. Consider the representation of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that the initial point of $\boldsymbol{b}$ is placed on the terminal point of $\boldsymbol{a}$. Then the vector $\boldsymbol{c}$ with initial point the initial point of $\boldsymbol{a}$, and terminal point the terminal point of $\boldsymbol{b}$ is called the **sum** or **resultant** of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and is denoted by $\boldsymbol{a} + \boldsymbol{b}$ (i.e., $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$). (Figure 3)

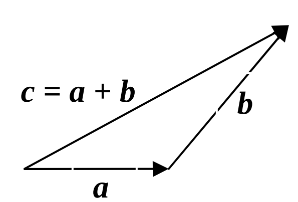


Figure 3

If $\boldsymbol{a} = \boldsymbol{a}'$ and $\boldsymbol{b} = \boldsymbol{b}'$ then $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{a}' + \boldsymbol{b}'$. This is so since the two triangles obtained are congruent. (Figure 4)

Figure 4

The addition of more than two vectors

The sum of more than two vectors is obtained by extension of the concept of addition defined above. Figure 5 indicates how the sum $\boldsymbol{e}$ of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ is found.

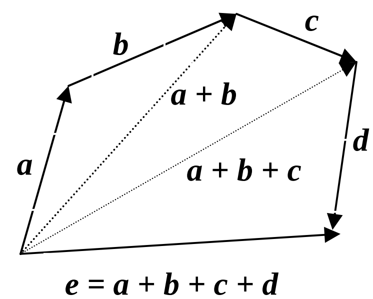


Figure 5

The sum of the vectors $\boldsymbol{a} = \overrightarrow{PS}$ and $\boldsymbol{b} = \overrightarrow{PQ}$ can be constructed as the diagonal of the parallelogram *PQRS*. This description of vector addition is called the **Parallelogram Law** of vector addition. (Figure 6)

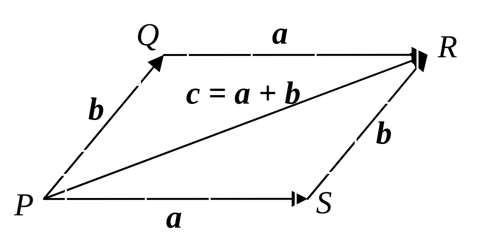


Figure 6

Subtraction of Vectors

The **difference** of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ denoted by $\boldsymbol{a} - \boldsymbol{b}$ is the vector $\boldsymbol{c}$ such that $\boldsymbol{c} + \boldsymbol{b} = \boldsymbol{a}$. Subtracting a vector from another is the same as adding the negative of the second to the first: $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$. (Figure 7)

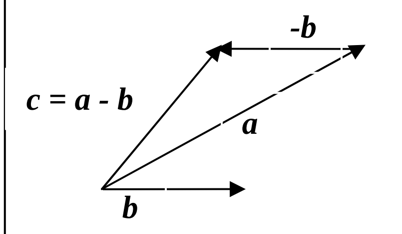


Figure 7

6. Properties of Vector Addition

For any vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ the following hold.

1. $\boldsymbol{a} + \mathbf{0} = \boldsymbol{a} = \mathbf{0} + \boldsymbol{a}$ (**0** is the additive identity)
2. $\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0} = (-\boldsymbol{a}) + \boldsymbol{a}$ (-$\boldsymbol{a}$ is the additive inverse of $\boldsymbol{a}$)
3. $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$ (Commutative Law for Addition)
4. $\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$ (Associative Law for Addition)

7. Multiplication of a Vector by a Scalar

Let s be a scalar and $\boldsymbol{a}$ a vector. Then the vector $s\boldsymbol{a}$ is defined to be the vector whose magnitude is $|s||\boldsymbol{a}|$ and which points in the same direction as $\boldsymbol{a}$ if s is positive, or in the opposite direction if s is negative. If $s = 0$, then $s\boldsymbol{a}$ is the null vector **0** whose magnitude is zero and which has no direction. For any scalar s, $s\boldsymbol{a}$ is called a **scalar multiple** of $\boldsymbol{a}$.

Examples:

(i) $2\boldsymbol{a}$ is the vector whose magnitude is twice that of $\boldsymbol{a}$ and is in the same direction as $\boldsymbol{a}$. (Figure 8a)

(ii) $-\frac{1}{2}\boldsymbol{a}$ is the vector whose magnitude is half that of $\boldsymbol{a}$ and is in the opposite direction to $\boldsymbol{a}$. (Figure 8b)

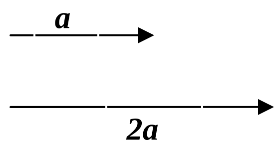


Figure 8a

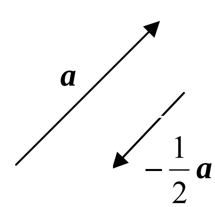


Figure 8b

8. Properties of Scalar Multiplication

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and any two scalars s and t the following hold.

(i) $0\boldsymbol{a} = \mathbf{0}$
(ii) $1\boldsymbol{a} = \boldsymbol{a}$
(iii) $(-1)\boldsymbol{a} = -\boldsymbol{a}$
(iv) $s(t\boldsymbol{a}) = (st)\boldsymbol{a} = t(s\boldsymbol{a})$
(v) $(s + t)\boldsymbol{a} = s\boldsymbol{a} + t\boldsymbol{a}$
(vi) $s(\boldsymbol{a}+\boldsymbol{b}) = s\boldsymbol{a} + s\boldsymbol{b}$

9. Properties of the Magnitude of a Vector

(i) $|\boldsymbol{a}| = 0$ if and only if $\boldsymbol{a} = \mathbf{0}$
(ii) $|\boldsymbol{a}| = |-\boldsymbol{a}|$
(iii) $|\boldsymbol{a} + \boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}|$ (Triangle Inequality)
(iv) $||\boldsymbol{a}| - |\boldsymbol{b}|| \leq |\boldsymbol{a} - \boldsymbol{b}|$

Note: The triangle inequality follows from the fact that the length of any side of a triangle does not exceed the sum of the lengths of the other two sides.

Proof of (iv):

$|\boldsymbol{a}| = |\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{b}| \leq |\boldsymbol{a} - \boldsymbol{b}| + |\boldsymbol{b}|$ (by the triangle inequality)
Therefore $|\boldsymbol{a}| - |\boldsymbol{b}| \leq |\boldsymbol{a} - \boldsymbol{b}|$.

$|\boldsymbol{b}| = |\boldsymbol{b} - \boldsymbol{a} + \boldsymbol{a}| \leq |\boldsymbol{b} - \boldsymbol{a}| + |\boldsymbol{a}|$ (by the triangle inequality)
Therefore $|\boldsymbol{b}| - |\boldsymbol{a}| \leq |\boldsymbol{b} - \boldsymbol{a}| = |\boldsymbol{a} - \boldsymbol{b}|$.

Thus $||\boldsymbol{a}| - |\boldsymbol{b}|| \leq |\boldsymbol{a} - \boldsymbol{b}|$.

10. Position Vectors

Let O be a fixed point in space (called the origin). If P is any point in space and $\overrightarrow{OP} = \boldsymbol{r}$, we say that the **position vector** of P with respect to the origin O is $\boldsymbol{r}$.

Note: When we speak of points with position vectors it is to be understood that all the vectors are expressed with respect to the same origin.

Result 1: If A and B are any two points with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, then $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$.

Proof:

$\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. Also $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$.

Therefore, $\boldsymbol{a} + \overrightarrow{AB} = \boldsymbol{b}$. Hence $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$.

(Figure 9)

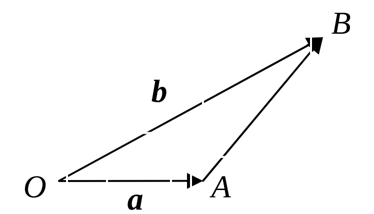


Figure 9

11. Vectors in a Plane

A Cartesian coordinate system consists of a fixed point O, called the origin, and two mutually perpendicular axes, Ox and Oy (or just x and y) with the same unit of length in both axes. Any point in a plane can be represented in a Cartesian coordinate system. This sets up a 1 – 1 correspondence between the points in the plane and ordered pairs of real numbers.

Consider a Cartesian coordinate system Oxy. Let $\boldsymbol{i}$ denote the unit vector parallel to the x axis in the positive direction, and let $\boldsymbol{j}$ denote the unit vector in the positive y direction. $\boldsymbol{i}, \boldsymbol{j}$ are called **base vectors** and the set $\{\boldsymbol{i}, \boldsymbol{j}\}$ is said to form a **basis** for the vectors in the plane. (Figure 10)

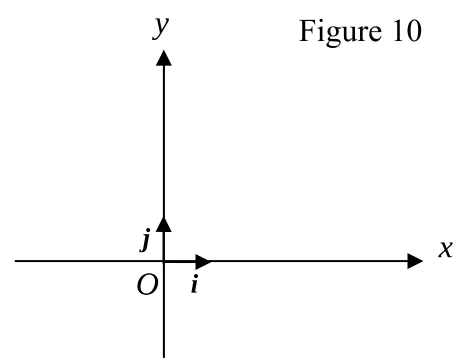


Figure 10

Any non-zero vector in the plane can be represented as a vector with initial point O and written uniquely as $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ (Figure 11). The numbers a_1 and a_2 are called the **components** of the vector $\boldsymbol{a}$ in the x and y directions respectively. The component of a vector in a given direction is the orthogonal projection of the vector in that direction. $a_1\boldsymbol{i}$ and $a_2\boldsymbol{j}$ are called the **component vectors** of $\boldsymbol{a}$ in the x and y directions respectively. If the point A has coordinates (a_1, a_2), then $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ is called the **position vector** of $A(a_1, a_2)$ with respect to the origin O. Vector $\boldsymbol{a}$ may also be written as the ordered pair (a_1, a_2).

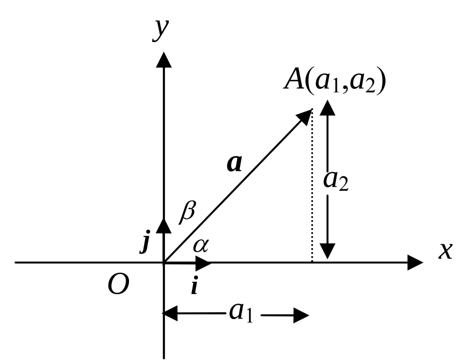


Figure 11

The distance of A from O can be determined by using Pythagoras' theorem. The distance from O to A, $|\overrightarrow{OA}| = |\boldsymbol{a}| = \sqrt{a_1^{\,2} + a_2^{\,2}}$ (Figure 11).

Suppose α is the angle between the vector $\boldsymbol{a}$ and the positive x direction.

Then $a_1 = |\boldsymbol{a}| \cos \alpha$, $a_2 = |\boldsymbol{a}| \sin \alpha$, and $\tan \alpha = \dfrac{a_2}{a_1}$.

If β is the angle between the vector $\boldsymbol{a}$ and the positive y direction, then α and β are called **direction angles**. The numbers $\cos \alpha$ and $\cos \beta$ are called **direction cosines** of the vector $\overrightarrow{OA}$. $\cos\alpha = \dfrac{a_1}{|\boldsymbol{a}|}$, $\cos\beta = \dfrac{a_2}{|\boldsymbol{a}|}$ in terms of components and, $\cos \beta = \sin \alpha$.

$\cos^2 \alpha + \cos^2 \beta = 1$ and $(\cos \alpha, \cos \beta)$ is a unit vector in the same direction as $\boldsymbol{a}$.

To determine the components of a vector, any directed line segment representing the vector can be used. If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points in the xy plane, the vector represented by the directed line segment P_1P_2 (initial point P_1 and terminal point P_2) is $(x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j}$. Any other directed line segment equivalent to P_1P_2 would give the same components (Figure 12).

Example:

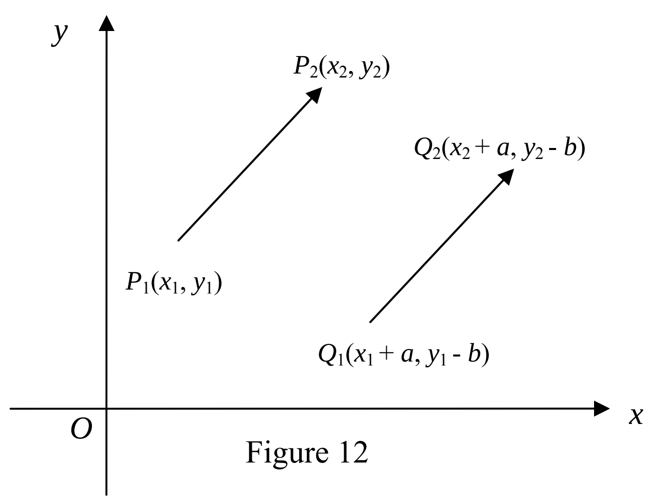


Figure 12

In Figure 12, the vector represented by the directed line segment P_1P_2 is $(x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j}$. The vector represented by the directed line segment Q_1Q_2 which is equivalent to P_1P_2 is $((x_2 + a) - (x_1 + a))\boldsymbol{i} + ((y_2 - b) - (y_1 - b))\boldsymbol{j} = (x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j}$. Thus the two equivalent directed line segments give the same components.

The magnitude of the vector $\overrightarrow{P_1P_2}$, $|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ and s is a scalar, then

(i) $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j}$

(ii) $\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1)\boldsymbol{i} + (a_2 - b_2)\boldsymbol{j}$

(iii) $s\boldsymbol{a} = sa_1\boldsymbol{i} + sa_2\boldsymbol{j}$

If the vectors are represented by ordered pairs $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2)$, then the above equations become

(i) $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, a_2 + b_2)$

(ii) $\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1, a_2 - b_2)$

(iii) $s\boldsymbol{a} = (sa_1, sa_2)$

12. Vectors in Three-Dimensional Space

As in the case of a point in a plane, any point in space can be represented in a Cartesian coordinate system which consists of a fixed point O, the origin, and three mutually perpendicular axes, Ox, Oy and Oz with the same unit of length along all three axes. The axes are placed in such a way that they form a right-handed set. This means that if a screw is placed at the origin and is turned in the sense from the positive x axis to the positive y axis, then it moves in the direction of the positive z axis.

A point A is located within this coordinate system by giving its directed distances from O in the directions of the positive x, y and z axes. Therefore, if the coordinates of A are (a_1, a_2, a_3), this means that A is a_1 units from O in the direction of Ox, a_2 units from O in the direction of Oy and a_3 units from O in the direction of Oz. The distance from O to A is $\sqrt{a_1^2 + a_2^2 + a_3^2}$.

Let $\boldsymbol{i}$ denote the unit vector parallel to the x axis in the positive direction, $\boldsymbol{j}$ the unit vector in the positive y direction and $\boldsymbol{k}$ the unit vector in the positive z direction respectively. $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are called **base vectors** and the set $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is said to form a **basis** for the vectors in space. Any non-zero vector $\boldsymbol{a}$ in space can be represented as a vector with initial point O and written uniquely as $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ (Figure 13). The numbers a_1, a_2 and a_3 are called the **components** (or rectangular components) of the vector $\boldsymbol{a}$ in the x, y and z directions respectively. $a_1\boldsymbol{i}$, $a_2\boldsymbol{j}$ and $a_3\boldsymbol{k}$ are called the **component vectors** (or rectangular component vectors) of $\boldsymbol{a}$ in the x, y and

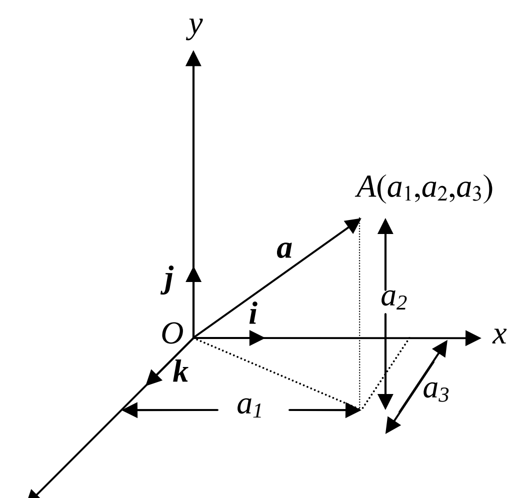


Figure 13

z directions respectively. If the point A has coordinates (a_1, a_2, a_3), then $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ is called the **position vector** of $A(a_1, a_2, a_3)$ with respect to the origin O and is denoted by $\overrightarrow{OA}$. Vector $\overrightarrow{OA} = \boldsymbol{a}$ may also be written as (a_1, a_2, a_3).

$$|\overrightarrow{OA}| = |\boldsymbol{a}| = \sqrt{a_1^{\ 2} + a_2^{\ 2} + a_3^{\ 2}}$$

Suppose α, β and γ are the angles between the vector $\boldsymbol{a}$ and the positive x, y and z directions respectively. Then $a_1 = |\boldsymbol{a}| \cos \alpha$, $a_2 = |\boldsymbol{a}| \cos \beta$ and $a_3 = |\boldsymbol{a}| \cos \gamma$. α, β and γ are called **direction angles**. The numbers $\cos \alpha$, $\cos \beta$ and $\cos \gamma$ are called **direction cosines** of the vector $\overrightarrow{OA}$. In terms of components the direction cosines are given by $\cos\alpha = \dfrac{a_1}{|\boldsymbol{a}|}$, $\cos\beta = \dfrac{a_2}{|\boldsymbol{a}|}$ and $\cos\gamma = \dfrac{a_3}{|\boldsymbol{a}|}$. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ and $(\cos\alpha, \cos\beta, \cos\gamma)$ is a unit vector in the same direction as $\boldsymbol{a}$.

If A and B are two arbitrary points in space with position vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ respectively, then

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) - (b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}) = (b_1 - a_1)\boldsymbol{i} + (b_2 - a_2)\boldsymbol{j} + (b_3 - a_3)\boldsymbol{k}.$$

The magnitude of the vector $\overrightarrow{AB}$ is $|\overrightarrow{AB}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$.

As in the two dimensional case, vector addition, vector subtraction and scalar multiplication proceed component wise.

If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ and s is a scalar, then

(i) $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j} + (a_3 + b_3)\boldsymbol{k}$

(ii) $\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1)\boldsymbol{i} + (a_2 - b_2)\boldsymbol{j} + (a_3 - b_3)\boldsymbol{k}$

(iii) $s\boldsymbol{a} = sa_1\boldsymbol{i} + sa_2\boldsymbol{j} + sa_3\boldsymbol{k}$

If the vectors are represented by $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$, then the above equations become

(i) $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

(ii) $\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$

(iii) $s\boldsymbol{a} = (sa_1, sa_2, sa_3)$

13. The angle between two vectors

Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{CD} = \boldsymbol{b}$ be any two vectors. Let $\overrightarrow{OA_1}$ and $\overrightarrow{OC_1}$ be respectively the representations of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with initial point O. The angle θ between $\overrightarrow{OA_1}$ and $\overrightarrow{OC_1}$ such that $0 \le \theta \le \pi$ is called the **angle between the vectors *a* and *b***. (Figure 14)

If $\theta = 0$ or $\theta = \pi$, the vectors are said to be **parallel**.
If $\theta = \dfrac{\pi}{2}$, then the vectors are said to be **perpendicular**.

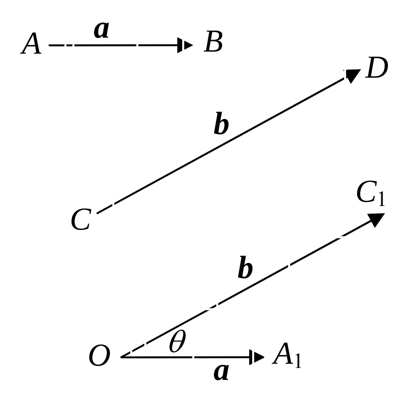


Figure 14

If A and B are two arbitrary points in space with position vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ respectively, then $\overrightarrow{AB} = (b_1 - a_1)\boldsymbol{i} + (b_2 - a_2)\boldsymbol{j} + (b_3 - a_3)\boldsymbol{k}$. Therefore, $AB^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2$.

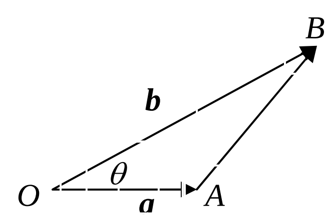


Figure 15

If θ is the angle between OA and OB, we have by applying the law of cosines to triangle OAB,

$AB^2 = OA^2 + OB^2 - 2\, OA.\, OB .\cos\, \theta.$
Therefore

$$\cos\theta = \frac{OA^2 + OB^2 - AB^2}{2OA.OB}$$

$$= \frac{(a_1{}^2 + a_2{}^2 + a_3{}^2) + (b_1{}^2 + b_2{}^2 + b_3{}^2) - \sum_{i=1}^{3}(b_i - a_i)^2}{2\sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}.\sqrt{b_1{}^2 + b_2{}^2 + b_3{}^2}}$$

$$= \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}.\sqrt{b_1{}^2 + b_2{}^2 + b_3{}^2}}.$$

Note: If A and B are two arbitrary points in a plane with position vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ respectively, and θ is the angle between OA and OB, then

$$\cos\theta = \frac{a_1b_1 + a_2b_2}{\sqrt{a_1{}^2 + a_2{}^2}.\sqrt{b_1{}^2 + b_2{}^2}}.$$

14. The Ratio Theorem

Let A and B be two given distinct points with position vectors $\vec{a}$ and $\vec{b}$ respectively. If the point R with position vector $\vec{r}$ divides the line segment AB in the ratio $m : n$ (with $m \neq 0,\ n \neq 0,\ m+n \neq 0$) then $\vec{r} = \dfrac{n\vec{a} + m\vec{b}}{n+m}$.

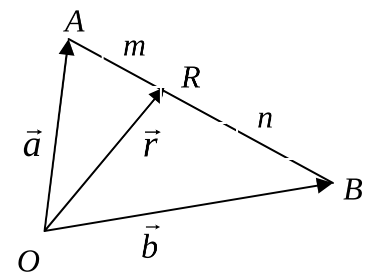


Figure 16

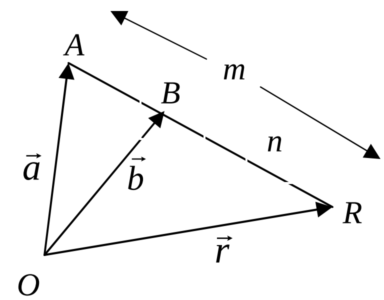


Figure 17

Proof:

If R divides AB in the ratio $m : n$, then $\dfrac{AR}{RB} = \dfrac{m}{n}$, or $nAR = mRB$.

(Here $\dfrac{m}{n}$ is positive or negative according as R divides AB internally, as in Figure16, or externally as in Figure 17).

Since AR and RB lie on the same line, $n\overrightarrow{AR} = m\overrightarrow{RB}$.
i.e., $n(\vec{r} - \vec{a}) = m(\vec{b} - \vec{r})$.
Therefore, $(n + m)\ \vec{r} = n\vec{a} + m\vec{b}$, and hence

$$\vec{r} = \frac{n\vec{a} + m\vec{b}}{n+m}.$$

Note: If R is the midpoint of AB, then $\boldsymbol{r} = \dfrac{\vec{a} + \vec{b}}{2}$.

15. Some Definitions and Results

1. Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if there exists a scalar t such that $\boldsymbol{a} = t\boldsymbol{b}$.

   If $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$, then $\boldsymbol{a} = t\boldsymbol{b}$ for some scalar t if and only if $(a_1, a_2, a_3) = (tb_1, tb_2, tb_3)$. i.e., $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $a_1 = tb_1$, $a_2 = tb_2$, $a_3 = tb_3$. i.e., $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3} = t$ (when $b_1 \neq 0,\ b_2 \neq 0,\ b_3 \neq 0$). Therefore in terms of components, **the condition for parallelism** is expressed by the equation $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$.

2. Suppose the two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.

If $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ and θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, then $\theta = \dfrac{\pi}{2}$ and $\cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}.\sqrt{b_1{}^2 + b_2{}^2 + b_3{}^2}} = 0$.

This gives us $a_1b_1 + a_2b_2 + a_3b_3 = 0$.

Therefore, in terms of components, the **condition for perpendicularity** is expressed by the equation $a_1b_1 + a_2b_2 + a_3b_3 = 0$.

3. Vectors lying on parallel straight lines or on one and the same straight line are called **collinear**. Two vectors are said to be **non-collinear** if they are not parallel to the same straight line.

4. Three distinct points A, B, C with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively are collinear (i.e., the three points A, B, C lie on the same straight line) if and only if there exists scalars x, y, z not all zero such that $x\vec{a} + y\vec{b} + z\vec{c} = \mathbf{0}$ and $x + y + z = 0$.

   Proof. Suppose the three distinct points A, B, C with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively are collinear. Suppose C divides AB in the ratio $m : n$. Then $\vec{c} = \dfrac{n\vec{a} + m\vec{b}}{n + m}$. Therefore, $n\vec{a} + m\vec{b} = (n + m)\vec{c}$.

   Let $x = n$, $y = m$ and $z = -(n + m)$.

   Then $x\vec{a} + y\vec{b} + z\vec{c} = \mathbf{0}$ and $x + y + z = 0$.

   Conversely, suppose there exists scalars x, y, z not all zero such that

   $x\vec{a} + y\vec{b} + z\vec{c} = \mathbf{0}$ and $x + y + z = 0$.

   Without loss of generality, let us assume that $z \neq 0$. Then $z = -(x + y)$ and therefore, $\vec{c} = \dfrac{x\vec{a} + y\vec{b}}{x + y}$. i.e., C divides AB in the ratio $y : x$ and A, B and C are collinear.

5. If a particle moves from an initial position (x_1, y_1, z_1) to another position (x_2, y_2, z_2), the **displacemen**t of the particle is the vector represented by the directed line segment extending from its initial position to its final position. This vector is $(x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j} + (z_2 - z_1)\boldsymbol{k}$. Thus if the initial position vector is $\overrightarrow{OR_1} = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ and the final position vector is $\overrightarrow{OR_2} = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$, then the displacement is $\overrightarrow{R_1R_2} = \overrightarrow{OR_2} - \overrightarrow{OR_1}$.

6. Vectors that are parallel to the same plane are said to be **coplanar**.

16. Scalar Product

The **scalar product** of the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, denoted by $\boldsymbol{a.b}$ (pronounced $\boldsymbol{a}$ dot $\boldsymbol{b}$) is the number $\boldsymbol{a.b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ where θ $(0 \le \theta \le \pi)$ denotes the angle between the vectors. The scalar product is also called the **dot product** or the inner product.

From Figure 18 we identify $|\boldsymbol{b}|\cos\theta$ as the component of $\boldsymbol{b}$ parallel to $\boldsymbol{a}$, that is, the length of the orthogonal projection of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$, with the appropriate sign.

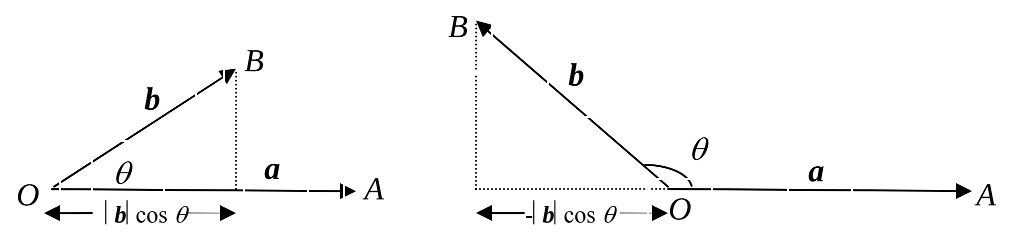


Figure 18

Thus we can interpret $\boldsymbol{a.b}$ as

(*length of* $\boldsymbol{a}$)(*signed component of* $\mathbf{b}$ *along* $\boldsymbol{a}$),

and since the definition is symmetric in $\boldsymbol{a}$ and $\boldsymbol{b}$, it can also be interpreted as

(*length of* $\mathbf{b}$)(*signed component of* $\mathbf{a}$ *along* $\mathbf{b}$).

It follows that the components of $\boldsymbol{a}$ in the x, y, z directions equal $\boldsymbol{a.i}$, $\boldsymbol{a.j}$ and $\boldsymbol{a.k}$ respectively.
Thus the vector $\boldsymbol{a}$ can be expressed as $\boldsymbol{a} = (\boldsymbol{a.i})\boldsymbol{i} + (\boldsymbol{a.j})\boldsymbol{j} + (\boldsymbol{a.k})\boldsymbol{k}$.

If $\boldsymbol{a} = \mathbf{0}$ or $\boldsymbol{b} = \mathbf{0}$, then $|\boldsymbol{a}| = 0$ or $|\boldsymbol{b}| = 0$ and hence $\boldsymbol{a.b} = 0$.
Also, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, then $\theta = \frac{\pi}{2}$ and hence $\boldsymbol{a.b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(\frac{\pi}{2}) = 0$ even if both vectors are non-zero.

Thus for the unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ of the coordinate axes we have
$\boldsymbol{i.i} = \boldsymbol{j.j} = \boldsymbol{k.k} = 1$ and $\boldsymbol{i.j} = \boldsymbol{j.k} = \boldsymbol{k.i} = 0$.

If $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are arbitrary vectors and s is a scalar, then

(i)	$\boldsymbol{a.b} = \boldsymbol{b.a}$	Commutative Law
(ii)	$\boldsymbol{a.}(\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a.b} + \boldsymbol{a.c}$	Distributive Law
(iii)	$\boldsymbol{a.}(s\boldsymbol{b}) = (s\boldsymbol{a})\boldsymbol{.b} = s(\boldsymbol{a.b})$	

If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, then using the above facts we obtain

$\boldsymbol{a}.\boldsymbol{b} = (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}).(b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}) = a_1b_1 + a_2b_2 + a_3b_3$.

In particular when $\boldsymbol{a} = \boldsymbol{b}$, we obtain
$\boldsymbol{a}.\boldsymbol{a} = a_1^2 + a_2^2 + a_3^2 = |\,\boldsymbol{a}\,|^2$
Thus $|\,\boldsymbol{a}\,| = \sqrt{\boldsymbol{a}.\boldsymbol{a}}$

17. <u>Vector Product</u>

The **vector product** of the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, denoted by $\boldsymbol{a} \times \boldsymbol{b}$ (pronounced $\boldsymbol{a}$ cross $\boldsymbol{b}$), is defined to be the vector $\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta\, \boldsymbol{n}$ where θ $(0 \le \theta \le \pi)$ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\boldsymbol{n}$ is a unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$, such that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{n}$ form a right handed system. (Figure 19)

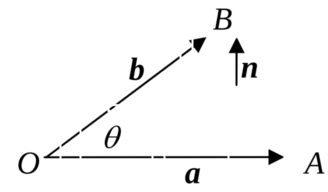


Figure 19

$\boldsymbol{a} \times \boldsymbol{b}$ is also called the **cross product** of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

We note that $|\,\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$ is the area of the parallelogram determined by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (computed as base times height). (Figure 20) Therefore the area of the triangle determined by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to $\frac{1}{2}\,|\,\boldsymbol{a} \times \boldsymbol{b}| = \frac{1}{2}\,|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$.

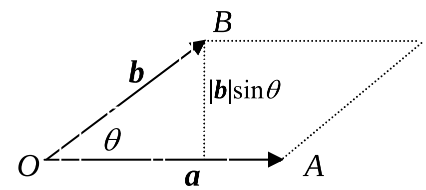
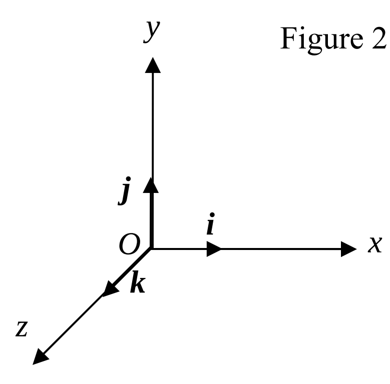


Figure 20

If $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$, then $|\,\boldsymbol{a}\,| = 0$ or $|\,\boldsymbol{b}\,| = 0$ and hence $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$.
Also, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel then $\theta = 0$ and hence $\boldsymbol{a} \times \boldsymbol{b} = |\,\boldsymbol{a}\,||\,\boldsymbol{b}\,| \sin 0\, \boldsymbol{n} = \boldsymbol{0}$.

Thus for the unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ of the coordinate axes we have

$\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{0}$
$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k},\ \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i},\ \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$
$\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j},\ \boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k},\ \boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i}$.

Figure 21


If $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are arbitrary vectors and s is a scalar, then

(i) $\boldsymbol{a} \times \boldsymbol{b} = -\, \boldsymbol{b} \times \boldsymbol{a}$

(ii) $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{a} \times \boldsymbol{c})$ Distributive Law

(iii) $\boldsymbol{a} \times (s\boldsymbol{b}) = (s\boldsymbol{a}) \times \boldsymbol{b} = s(\boldsymbol{a} \times \boldsymbol{b})$

Note: The commutative and associative laws fail for the vector product. i.e.,

(i) $\boldsymbol{a} \times \boldsymbol{b} \neq \boldsymbol{b} \times \boldsymbol{a}$

(ii) $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} \neq \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$

If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, then using the above facts we obtain

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} &= (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) \times (b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}) \\ &= (a_2b_3 - a_3b_2)\boldsymbol{i} + (a_3b_1 - a_1b_3)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}\end{aligned}$$

This may conveniently be written in determinant form:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

18. <u>Triple Scalar Product</u>

The **triple scalar product** of the three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, denoted by $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$, is defined by $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = \boldsymbol{a} \,.\, (\boldsymbol{b} \times \boldsymbol{c})$

If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$, $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ and $\boldsymbol{c} = c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k}$,
then

$$\begin{aligned}[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] &= \boldsymbol{a}.\, (\boldsymbol{b} \times \boldsymbol{c}) \\ &= (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}).\, [(b_2c_3 - b_3c_2)\boldsymbol{i} + (b_3c_1 - b_1c_3)\boldsymbol{j} + (b_1c_2 - b_2c_1)\boldsymbol{k}] \\ &= a_1(b_2c_3 - b_3c_2) + a_2\,(b_3c_1 - b_1c_3) + a_3\,(b_1c_2 - b_2c_1) \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\end{aligned}$$

Therefore, $[\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

The modulus of the triple scalar product $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$ is equal to the volume of the parallelepiped constructed with $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ as co-terminal edges as in Figure 22.

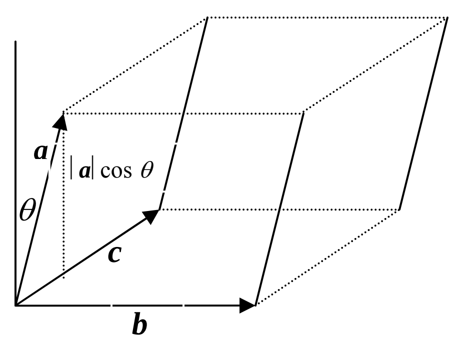


Figure 22

The base of the solid is a parallelogram whose area is given by $|\boldsymbol{b} \times \boldsymbol{c}|$. The height of the parallelepiped is the length of the component of $\boldsymbol{a}$ perpendicular to the base, which equals the magnitude of $|\boldsymbol{a}| \cos\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b} \times \boldsymbol{c}$. i.e., the volume of the parallelepiped with co-terminal edges $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ is the modulus of $|\boldsymbol{a}| \cos\theta\, |\boldsymbol{b} \times \boldsymbol{c}|$, which is the modulus of $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = \boldsymbol{a}.\ (\boldsymbol{b} \times \boldsymbol{c})$

From the properties of determinants, it follows that $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = [\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a}] = [\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b}]$

The sign changes whenever two of the vectors are interchanged; for example,
$[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = - [\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{b}]$

Also, the position of the dot and the cross can be changed freely; for example,
$\boldsymbol{a}.\ (\boldsymbol{b} \times \boldsymbol{c}) = [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = [\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b}] = \boldsymbol{c}.\ (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{a} \times \boldsymbol{b}).\ \boldsymbol{c}$
i.e., $\boldsymbol{a}.\ (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}).\ \boldsymbol{c}$

The triple scalar product is also linear in each of its factors; i.e.,
$[s\boldsymbol{a} + \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}] = s[\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}] + [\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}]$
$[\boldsymbol{a}, s\boldsymbol{b} + \boldsymbol{c}, \boldsymbol{d}] = s[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}] + [\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}]$
$[\boldsymbol{a}, \boldsymbol{b}, s\boldsymbol{c} + \boldsymbol{d}] = s[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] + [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}]$

Clearly, if any two of the vectors are equal, then the triple scalar product will be zero.

The volume of a tetrahedron with $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ as co-terminal edges (as in Figure 23) is obtained as follows:

$$\text{Volume} = \frac{1}{3} \times (\text{area of base}) \times (\text{height})$$
$$= \frac{1}{3} \left| \left(\frac{1}{2} |\boldsymbol{a} \times \boldsymbol{b}|\right) |\boldsymbol{c}| \cos\theta \right| = \frac{1}{6} |(\boldsymbol{a} \times \boldsymbol{b}).\ \boldsymbol{c}|$$

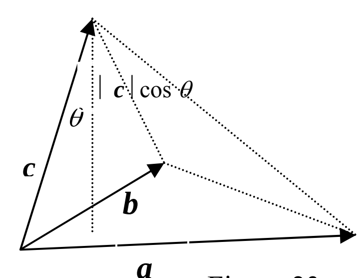


Figure 23

Result:
Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be three non-zero, non-parallel vectors. Then $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are coplanar if and only if $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = 0$.

Proof: If $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = 0$, then the vector $\boldsymbol{a}$ is perpendicular to the vector $\boldsymbol{b} \times \boldsymbol{c}$. Therefore $\boldsymbol{a}$ lies on the plane containing the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ or in a plane parallel to this plane. Hence $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are coplanar. On the other hand, if $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are coplanar, then $\boldsymbol{b} \times \boldsymbol{c}$ is perpendicular to $\boldsymbol{a}$ and hence $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = 0$.

19. Vector Identities

(i) $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}.\boldsymbol{b})\boldsymbol{c}$
(ii) $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b}.\boldsymbol{c})\boldsymbol{a}$
(iii) $(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = [\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}]\boldsymbol{b} - [\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}]\boldsymbol{a}$
(iv) $(\boldsymbol{a} \times \boldsymbol{b})\ .\ (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a}.\boldsymbol{c})(\boldsymbol{b}.\boldsymbol{d}) - (\boldsymbol{a}.\boldsymbol{d})(\boldsymbol{b}.\boldsymbol{c})$