IT3303: Mathematics for Computing II

Matrices

1.1 Definition of a Matrix

A matrix is an array of m × n elements arranged in m rows and n columns. Such a matrix A is usually denoted by

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots\dots & \dots & \dots & .. \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m\times n} = [a_{ij}]_{m\times n}$$

Where $a_{11,}a_{12},.........a_{mn}$ are called the **elements** of the matrix

The element a_{ij} is called the ijth entry of the matrix and it appears in the ith row and the jth column of the matrix. A matrix with m rows and n columns is called an **m x n (**read **m by n)** matrix and we say that the matrix is of **order** m x n. We often denote matrices by capital letters

1.2 Column and row matrices

If a matrix A is such that A consists of just one row, then A is said to be a **row matrix**

For example

$$\begin{pmatrix} 2 & 4 & 7 \end{pmatrix}_{1\times 3}$$

is a row matrix with three elements.

Note: In a row matrix, m=1

If a matrix A is such that A consists of just one column, then A is said to be a **column matrix**.

For example $\begin{pmatrix} -3 \\ 6 \\ 90 \end{pmatrix}_{3\times 1}$

is a column matrix with three elements.

Note: In a column matrix, n=1

1.3 Different Types of Matrices and their Properties

1.3.1 Square Matrix

If in a matrix A, the number of rows equals the number of columns, then A is said to be a **square matrix**. If the number of rows in a square matrix is n, then A is called a matrix of order n.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}$$

is a square matrix of order 2.

1.3.2 Diagonal Matrix

Let A = (a_{ij}) be a square matrix of order n. Then A is said to be a **diagonal matrix** if a_{ij} = 0 whenever i ≠ j, where i, j ∈ {1, 2,….. n}. The elements a_{ii} ,where i ∈ {1, 2,….. n} are called diagonal elements.

For example

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3\times 3}$$

is a diagonal matrix of order 3.

Note that the diagonal elements of a diagonal matrix may also be zero.

1.3.3 Null or Zero Matrix

Let A = (a_{ij}) be an m x n matrix. Then A is said to be a **null or zero matrix** if a_{ij} = 0 for all i ∈ {1, 2,.. m}, j ∈ {1,2,...n}.

For example

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a null matrix of order 2 x 3.

1.3.4 Symmetric Matrix

An n x n matrix A = (a_{ij}) is called a **symmetric matrix** if $a_{ij} = a_{ji}$ for all i, j ∈ {1, 2, ….n}.

For example

$$\begin{pmatrix} 4 & 2 & 0 & 1 \\ 2 & 6 & 5 & -2 \\ 0 & 5 & 0 & 1 \\ 1 & -2 & 1 & 9 \end{pmatrix}_{4\times 4}$$

is symmetric.

Note: a symmetric matrix is a square matrix.

1.3.5 Skew-symmetric Matrix

An n x n matrix A = (a_{ij}) is called a **skew-symmetric matrix** if $a_{ij} = - a_{ji}$ for all i, j ∈ {1, 2, ….n}.

For example

$$\begin{pmatrix} 0 & 2 & -4 \\ -2 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix}_{3\times 3}$$

is skew-symmetric.

1.3.6 Upper Triangular Matrix

Let A be an n x n square matrix such that all the entries below the diagonal are zero; i.e. $a_{ij} = 0$ whenever i > j, where i, j ∈ {1, 2,….n}. Then A is said to be an **upper triangular matrix**.

For example

$$\begin{pmatrix} 0 & 5 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3\times 3}$$

is a 3 x 3 upper triangular matrix.

Note that the diagonal elements of an upper triangular matrix need not be zero.

1.3.7 Lower Triangular Matrix

Let A be an n x n square matrix such that all the entries above the diagonal are zero; i.e. $a_{ij} = 0$ whenever i < j, where i, j $\in$ {1, 2,….n}. Then A is said to be a **lower triangular matrix**.

For example
$$\begin{pmatrix} -4 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix}_{3\times3}$$

is a 3 x 3 lower triangular matrix.

Note that the diagonal elements of a lower triangular matrix need not be zero.

1.3.8 Identity Matrix

Suppose A is an n x n diagonal matrix such that all the diagonal elements are equal to 1. Then A is said to be an identity **matrix** of order n.

For example
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times3}$$

is an identity matrix of order 3.

Note that identity matrix is a square matrix.

1.3.9 Equality of Matrices

Two matrices A = (a_{ij}) and B = (b_{ij}) are said to be **equal** if A and B have the same order, say m x n, and if $a_{ij} = b_{ij}$ for all i $\in$ {1, 2, …, m}, for all j $\in$ {1, 2, …, n}.

For example

$$\begin{pmatrix} 3 & 4 & 9 \\ 16 & 25 & 64 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 9 \\ 16 & 25 & 64 \end{pmatrix}$$

1.4 Matrix Addition

Let A = (a_{ij}) and B = (b_{ij}) be two matrices having the same order, say m x n. We define the sum of A and B denoted by A + B to be the matrix C = (c_{ij}) where $c_{ij} = a_{ij} + b_{ij}$ for all i ∈ {1, 2,….m} and j ∈ {1, 2,….n}.

Example.

Let $A = \begin{pmatrix} 1 & 0 & 8 \\ 2 & -1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 & 0 \\ -5 & -4 & 1 \end{pmatrix}$ Then

$$A + B = \begin{pmatrix} 4 & 2 & 8 \\ -3 & -5 & 5 \end{pmatrix}$$

1.5 Scalar Multiplication of a Matrix

Let A = (a_{ij}) be a m x n matrix . The **product** of the **scalar** k and the matrix A, denoted by k.A (or kA) is the matrix B = (b_{ij}) where $b_{ij} = k\,a_{ij}$ for all i ∈ {1, 2,….m} and j ∈{1, 2,….n}.

Example. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ and k = -2 then

$$kA = \begin{pmatrix} -2 & -4 & -6 \\ -8 & -10 & -12 \\ -14 & -16 & -18 \end{pmatrix}$$

Notation: We write

- -A for -1×A and
- A – B for A + (-B)

Results:

Let A, B and C be three matrices of the same order. Then the following properties hold.

- A + B = B + A.
- A + (B + C) = (A + B) + C.

- $(k_1k_2)A = k_1(k_2A)$.
- $(k_1 + k_2)A = k_1A + k_2A$.
- $k(A + B) = kA + kB$.
- $1.A = A$
- $0.A_{m \times n} = 0_{m \times n}$

1.6 Matrix Multiplication and its Properties

Let A = (a_{ij}) be a 1 x n row matrix and B = (b_{ij}) be a n x 1 column matrix. Then we define the product of the row matrix A and the column matrix B by

$$AB = (a_{11}, a_{12}, \ldots\ldots\ldots a_{1n}) \times \begin{pmatrix} b_{11} \\ b_{21} \\ . \\ . \\ b_{n1} \end{pmatrix} = \left[a_{11}b_{11} + a_{12}b_{21} + \ldots\ldots\ldots + a_{1n}b_{n1}\right]_{1\times 1}$$

Now let A = $(a_{ij})_{m \times p}$ and B =$(b_{ij})_{p \times n}$ be any two matrices of order m x p and p x n respectively. Then the product AB is defined as the matrix C of order m x n whose ijth entry is obtained by multiplying the ith row of A by the jth column of B. That is, if C = $(c_{ij})_{m \times n}$,

$$c_{ij} = (a_{i1}, a_{i2}, \ldots\ldots\ldots a_{ip}) \times \begin{pmatrix} b_{1j} \\ b_{2j} \\ . \\ . \\ b_{pj} \end{pmatrix} = [\, a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots\ldots\ldots + a_{ip}b_{pj} \,]$$

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

Note: The product of two matrices A and B is defined only when the number of columns of A is equal to the number of rows of B.

Example:

Let A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}$ and B = $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix}_{2\times 3}$ then

$$AB = \begin{pmatrix} 1.1+2.2 & 1.-1+2.0 & 1.0+2.1 \\ 3.1+4.2 & 3.-1+4.0 & 3.0+4.1 \end{pmatrix}_{2\times 3}$$

$$= \begin{pmatrix} 5 & -1 & 2 \\ 11 & -3 & 4 \end{pmatrix}_{2\times 3}$$

Note: Matrix multiplication is not commutative. That is, in general, AB ≠ BA.

Results : Let A = (a_{ij}), B = (b_{ij}) and C = (c_{ij}), be three matrices and let k be a constant. Then,

- if A is m x n and B is n x p, then k(AB) = (kA)B = A(kB).
- if B and C are m x n and A is n x p, then (B + C)A = BA + CA.
- if A is m x n and B and C are n x p, then A(B + C) = AB + AC.
- if A is m x n, B is n x p and C is p x q, then A(BC) = (AB)C.
- if A is an n x m matrix, and I is the n x n identity matrix, then IA = A. Also if I is the m x m identity matrix, then AI = A.

1.7 Transpose of a Matrix and Orthogonal Matrix

1.7.1 Transpose of a Matrix

Let A = (a_{ij}) be an m x n matrix. Then the **transpose of A** denoted by A^T is the n x m matrix (b_{ij}) where $b_{ij} = a_{ji}$ for all i ∈ {1, 2,…., n}, j ∈ {1, 2,….., m}.

Example:

Let A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ the A^T = $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$

Results: Let A = (a_{ij}), B = (b_{ij}) be m x n matrices, and let C = (c_{ij}) be an n x p matrix. Then,

- $(A + B)^T = A^T + B^T$
- $(A^T)^T = A$
- $(AC)^T = C^T A^T$

1.7.2 Orthogonal Matrix

A square matrix A = (a_{ij}) is said to be an **Orthogonal matrix** if

$$AA^T = A^T A = I$$

For example

$$\begin{bmatrix} Cos\theta & -Sin\theta \\ Sin\theta & Cos\theta \end{bmatrix}$$

is an orthogonal matrix of order 2.

1.7.3 Invertible Matrices

Let A be an n x n square matrix. We say that A is **invertible** if there exists a n x n matrix B such that AB = BA = $I_{n \times n}$ where $I_{n \times n}$ is the n x n identity matrix.

Example:

Let A = $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ then

AB = I and BA = I

We call B the **inverse** of A and denote it by A^{-1}

Note:

1. If B is an inverse of A, then A is also an inverse of B

   If A^{-1}=B, then B^{-1} =A

2. Inverse of a matrix is unique
3. Every square matrix is not invertible
4. $(A^{-1})^{-1} = A$
5. $(AB)^{-1} = B^{-1}A^{-1}$
6. If A is an invertible diagonal matrix with diagonal elements a_{ij}, then A^{-1} is also a diagonal matrix with diagonal elements $1/a_{ij}$
7. $I^{-1} = I$

1.8 Determinants

Every square matrix A is associated with a scalar called the **determinant of A**, and is denoted by |A|.

Let A = (a_{ij}) be a square matrix of order one. Then we define **|A| =** a_{11}

1.8.1 Determinants of matrices of order two

Let A = (a_{ij}) be a square matrix of order two, i.e.,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Then we define **|A| =** $a_{11}a_{22} - a_{12}a_{21}$

Example:

Let A = $\begin{pmatrix} 5 & 1 \\ 2 & -3 \end{pmatrix}_{2\times 2}$

Then $|A|$ = 5 × (-3) – 1 × 2 = -17

1.8.2 Determinants of matrices of order three

Let A = (a_{ij}) be a square matrix of order three, i.e.,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then we define

$$|A| = a_{11}m_{11} - a_{12}m_{12} + a_{13}m_{13}$$

where m_{ij} is the determinant of the matrix of order 2 obtained by deleting the row and column containing a_{ij} .

Example:

Let A = $\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$

$$|A| = 1 \text{ x } \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} - 0 \text{ x } \begin{vmatrix} 2 & 2 \\ -1 & -1 \end{vmatrix} + 2 \text{ x } \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix}$$

$|A| = 1 \times (-1) - 0 \times (0) + 2 \times (1)$

$= 1$

1.8.3 Properties of Determinants

- Let A be a matrix of order n. Then $|A^T| = |A|$.

Eg.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -2$$

- Let A be a square matrix of order n. Then $|kA| = k^n|A|$ where k is a scalar.

Eg.

$$\left| 2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right| = \begin{vmatrix} 2 & 4 \\ 6 & 8 \end{vmatrix} = -8 = 2^2 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

- Let A be a square matrix of order n. If any two rows (or columns) of A are identical, then $|A| = 0$

Eg.

$$\begin{vmatrix} 1 & 3 & 1 \\ 2 & 4 & 2 \\ 3 & 5 & 3 \end{vmatrix} = 0$$

- If A = (a_{ij}) is a diagonal matrix or a triangular matrix of order n, then

$$|A| = a_{11}a_{22}.........a_{nn}$$

Eg.

$$\begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 6 \end{vmatrix} = 1.2.6 = 12$$

- Let I be the identity matrix of order n. Then $|I| = 1$

Eg. Let $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$| I | = 1(1-0) - 0(0 - 0) + 0(0-0) = 1$$

- Let A be a square matrix of order n. If B is obtained from A by interchanging any two rows (or columns) of A, then |B| = - |A|.

  Eg.

$$\begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 6 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 3 \\ 0 & 0 & 6 \\ 0 & 2 & 8 \end{vmatrix}$$

- Let A be a square matrix of order n. If B is obtained from A by multiplying a row (or column) by a nonzero scalar k, then |B| = k|A|

  Eg.

$$\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix}$$

- Let A be a square matrix of order n. If B is obtained from A by adding a scalar multiple of a row (or column) of A to another row (or column) of A, then |B| = |A|.

  Eg.

$$\begin{vmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 6 \end{vmatrix} = \begin{vmatrix} 1+0.k & 1+2k & 4+5k \\ 0 & 2 & 5 \\ 0 & 0 & 6 \end{vmatrix}$$

- If any row (or column) of a square matrix A is the sum of two or more elements, then the determinant can be expressed as the sum of two or more determinants.

  Eg.

$$\begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1+3 \\ 2 & 3 & 2+3 \\ 3 & 5 & 4+2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 5 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \\ 3 & 5 & 2 \end{vmatrix}$$

- Let A be a square matrix of order n. If B is also a square matrix of order n, then

$$|AB| = |A||B|.$$

1.9 Singular and Non-singular Matrices

If $|A| \neq 0$, then A is said to be a **nonsingular** matrix; otherwise it is said to be **singular**.

Note:

If A^{-1} exists, $|AA^{-1}| = |A||A^{-1}| = 1$

i.e., $|A^{-1}| = 1/|A|$

That is, If A is invertible, it is non-singular.

Example:

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then $|A| = 4 - 6 = -2 \neq 0$.

Therefore, A^{-1} exists and

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

1.10 Adjoint of a Square Matrix

Let $A = (a_{ij})$ be a square matrix of order n. The **cofactor** c_{ij} of a_{ij} is defined as

$$c_{ij} = (-1)^{i+j} m_{ij}$$

where m_{ij} is the determinant of the matrix of order n-1 obtained by deleting the row and column containing a_{ij}.

Example:

Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$ then then the **cofactor** c_{23} of a_{23} is derived as:

$c_{23} = (-1)^{2+3} m_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} = (1)(0) – (0)(-1) = 0$

Similarly, we can find the cofactors of all the elements a_{ij} of the matrix A and the cofactor matrix C of A can be given as:

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

Let A = (a_{ij}) be a square matrix of order n and let C denote its matrix of cofactors. The **adjoint of A** denoted by adj A, is C^T, the transpose of the matrix of cofactors.

Example:

Let A = $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ Then C = $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

Therefore adj A = C^T = $\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -1 & 0 & 1 \end{pmatrix}$

1.11 Finding the Inverse of a Matrix

If A is a non-singular matrix, then

$$A^{-1} = \frac{1}{|A|} \text{ adj A}$$

Example:

Let A = $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$

Then |A| = 1(2 – 0) – 0 + (-1)(0 – 1) = 3. Thus A is invertible

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

1.12 Systems of Linear Equations

A system of **m** linear equations in **n** unknowns is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots\dots\dots\dots a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots\dots\dots a_{2n}x_n &= y_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots\dots\dots a_{mn}x_n &= y_m \end{aligned}$$

where $y_1, y_2, \dots\dots y_m$ and a_{ij} $1 \le i \le m$, $1 \le j \le n$ are real numbers and $x_1, x_2, \dots\dots x_n$ are n unknowns.

Note: If $y_1 = y_2 = \dots\dots = y_m = 0$, the system is called a **homogeneous** system.

We write this system in matrix form as:

$$\begin{pmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_m \end{pmatrix}$$

$$AX = Y$$

A is called the **matrix of coefficients** of the system.

Note: if Y is zero (zero matrix), the system is called a **homogeneous** system.

1.12.1 Characteristics of the systems of Linear Equation

1. A solution of the system of linear equations is a set of values $x_1, x_2, \ldots\ldots x_n$ which satisfy the above m equations.
2. If the equations are homogeneous then, $x_1 = x_2 = \ldots\ldots = x_m = 0$ is a solution of the system.
3. If the system is not homogeneous, it is possible that no set of values will satisfy all the equations in the system. If this is the case the system is said to be **inconsistent**.
4. If there exists a solution which satisfies all the equations of the system, the system is said to be **consistent**.
5. A homogeneous system is always consistent since it has the trivial **solution** $x_1 = x_2 = \ldots\ldots = x_m = 0$
6. There are two possible types of solutions to a consistent system of linear equations. Either the system will have a unique solution, or it will have infinitely many solutions.
7. If a homogeneous system has a unique solution then, since the trivial solution is always a solution, the trivial solution will be its unique solution.

Examples:

(1)

$$2x + y = 5$$
$$x - y = 4.$$

This system has a unique solution, x = 3, y = -1.

(2)

$$2x + 3y + 4z = 5$$
$$x + 6y + 7z = 3$$

This system has infinitely many solutions of the form

x = k,
y = (10k - 23) / 3
z = (7 – 3k) where k is any scalar.

(3)

$$\begin{aligned} x + 2y - 3z &= -1 \\ 3x - y + 2z &= 7 \\ 5x + 3y - 4z &= 2 \end{aligned}$$

This system has no solution.

1.12.2 Elimination Method

The most fundamental method of finding solutions of systems of linear equations is the method of elimination.

Consider the following system.

$$\begin{aligned} 2x + 3y &= 3 \\ x - y &= 4. \end{aligned} \qquad \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

- We first eliminate y by multiplying the second equation by 3 and adding it to the first equation.

$$\begin{aligned} 5x + 0y &= 15 \\ x - y &= 4. \end{aligned} \qquad \begin{pmatrix} 5 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 4 \end{pmatrix}$$

- Now multiplying the first row by 1/5 we obtain

$$\begin{aligned} x + 0y &= 3 \\ x - y &= 4 \end{aligned} \qquad \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

- By multiplying the first row by –1 and adding it to row 2 we obtain

$$\begin{aligned} 1x + 0y &= 3 \\ 0x + (-1)y &= 1 \end{aligned} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

- Finally by multiplying the second row by -1, we obtain

$$\begin{aligned} 1x + 0y &= 3 \\ 0x + 1y &= -1 \end{aligned} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Note that the process of solving the system of this two linear equation stops when the coefficient matrix becomes identity matrix.

From this we can directly obtain the solution,

x = 3
y = -1

We have used two operations here, namely

1. addition of a scalar multiple of a row to another row
2. multiplication of a row by a scalar.

We use a similar method to find the solution of any system of linear equations. We apply 3 types of operations on the equations of the system to reduce them to another system of linear equation from which it will be possible to determine whether a system is consistent or not and if consistent to determine its solution. The operations we use are called **elementary row operations** and these are performed on the matrix of coefficients. The three operations are as follows:

1. Any two rows of a matrix may be interchanged.

2. A row may be multiplied by a nonzero constant

3. A multiple of one row may be added to another row

Example:

Consider the following system of three linear equations.

x + 2y –3z = -1
3x – y + 2z = 7
5x + 3y – 4z = 2

This system in matrix form

$$\begin{pmatrix} 1 & 2 & -3 \\ 3 & -1 & 2 \\ 5 & 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$$

Step 1: Multiplying the first row by –3 and adding it to the second row we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -7 & 11 \\ 5 & 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \\ 2 \end{pmatrix}$$

Step 2: Multiplying the first row by –5 and adding it to the third row we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -7 & 11 \\ 0 & -7 & 11 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \\ 7 \end{pmatrix}$$

Step 3: Multiplying row 2 by –1 and adding to row 3 we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -7 & 11 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \\ -3 \end{pmatrix}$$

Thus the system reduces to

$$\begin{aligned} x + 2y - 3z &= -1 \\ -7y + 11z &= 10 \\ 0 &= -3 \end{aligned}$$

This shows that the system is **inconsistent** since the third equation is false. Thus this system has no solution.

Example:

Consider the following system of linear equations.

$$\begin{aligned} x + 2y - 3z &= 6 \\ 2x - y + 4z &= 2 \\ 4x + 3y - 2z &= 14 \end{aligned}$$

This set of equations in matrix form

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 14 \end{pmatrix}$$

Step 1: Multiplying row 2 by –2 and adding to row 3 we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 0 & 5 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 10 \end{pmatrix}$$

Step 2: Multiplying row 1 by –2 and adding to row 2 we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 10 \\ 0 & 5 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \\ 10 \end{pmatrix}$$

Step 3: Adding row 2 to row 3 we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 10 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \\ 0 \end{pmatrix}$$

Step 4: Multiplying row 2 by -1/5 we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix}$$

Step 5: Multiplying row 2 by –2 and adding to row 1 we obtain

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

Thus the system reduces to

$x + z = 2$
$y - 2z = 2$
$0 = 0$

This system is consistent and has infinitely many solutions given by

x = k
y = 6 – 2k
z = 2 – k where k is a scalar.

Example:

Consider the following system of linear equations.

2x + y + 3z = 5
3x – 2y + 2z = 5
5x – 3y – z = 16

This set of equations in matrix form

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & -2 & 2 \\ 5 & -3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 16 \end{pmatrix}$$

Step 1: Adding row 1 to row 2 we obtain

$$\begin{pmatrix} 2 & 1 & 3 \\ 5 & -1 & 5 \\ 5 & -3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 16 \end{pmatrix}$$

Step 2: Multiplying row 2 by –1 and adding to row 3 we obtain

$$\begin{pmatrix} 2 & 1 & 3 \\ 5 & -1 & 5 \\ 0 & -2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 6 \end{pmatrix}$$

Step 3: Multiplying row 2 by 1/5 and then multiplying row 3 by –1/2 we obtain

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & -1/5 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}$$

Step 4: Multiplying row 3 by –1 and adding to row1 we obtain

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & -1/5 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ -3 \end{pmatrix}$$

Step 5: Multiplying row 1 by ½ we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1/5 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$$

Step 6: Multiplying row 1 by –1 and adding to row 2 and then multiplying row 3 by 1/5 and adding to row 2 we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 8/5 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -13/5 \\ -3 \end{pmatrix}$$

Step 7: Multiplying row 2 by 5/8 we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -13/8 \\ -3 \end{pmatrix}$$

Step 8: Multiplying row 2 by -3 and adding to row 3 we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -13/8 \\ 15/8 \end{pmatrix}$$

Step 9: Finally interchanging row 2 and row 3 we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 15/8 \\ -13/8 \end{pmatrix}$$

Thus the system is reduced to

x = 4
y = 15/8
z = -13/8

This is the unique solution to the system.

Result:

Suppose a system of linear equations in matrix form is AX = Y. If the matrix A is invertible, the system has a unique solution given by $X = A^{-1}Y$.

Consider the following system of linear equations.

x –z = 3
y + z = 3
x + 2z = 6.

In matrix form,

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}$$

The matrix A = $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ has $|A| \neq 0$

Thus A^{-1} must exist

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

Thus the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}$$

Therefore we obtain the unique solution

x = 4
y = 2
z = 1

Calculating the Inverse by Elimination

This method involves augmenting the square matrix A with the Identity matrix. The augmented matrix is designed by the symbol

A | I

Row operations are then employed to obtain an identity matrix on the left side of the vertical line

Concurrent with the identity matrix on the left of the vertical line, the inverse matrix is obtained on the right of the vertical line

This method transforms the augmented matrix, A | I, to inverse matrix I | A^{-1}

Example:

Let $A = \begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix}$ Then the augmented matrix can be written as

$$A \mid I = \left[\begin{array}{cc|cc} 5 & 3 & 1 & 0 \\ 6 & -2 & 0 & 1 \end{array}\right]$$

Step 1: Multiply row 1 by 1/5

$$\left[\begin{array}{cc|cc} 1 & 3/5 & 1/5 & 0 \\ 6 & -2 & 0 & 1 \end{array}\right]$$

Step 2: Multiply row 1 by -6 and add to row 2

$$\left[\begin{array}{cc|cc} 1 & 3/5 & 1/5 & 0 \\ 0 & -28/5 & -6/5 & 1 \end{array}\right]$$

Step 3: Multiply row 2 by -5/28

$$\left[\begin{array}{cc|cc} 1 & 3/5 & 1/5 & 0 \\ 0 & 1 & 6/28 & -5/28 \end{array}\right]$$

Step 4: Multiply row 2 by -3/5 and add to row 1

$$\left[\begin{array}{cc|cc} 1 & 0 & 2/28 & 3/28 \\ 0 & 1 & 6/28 & -5/28 \end{array}\right]$$

$$\underbrace{\qquad\quad}_{\text{I}} \qquad \underbrace{\qquad\qquad\qquad}_{\text{A}^{-1}}$$

Therefore,

$$A^{-1} = \begin{bmatrix} 2/28 & 3/28 \\ 6/28 & -5/28 \end{bmatrix}$$